SU$(2)_L \times U(1)_Y$ 対称な Lagrangian から EWSB後の SM Lagrangian の導出

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$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}^{\text{kin}}_{\text{lepton}} + \mathcal{L}^{\text{mass}}_{\text{lepton}} + \mathcal{L}^{\text{kin}}_{\text{quark}} + \mathcal{L}^{\text{mass}}_{\text{quark}}$$

$$j, k = 1 \sim 3 \qquad L_L = \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix} \qquad Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

	T	T^3	Y	$Q = T^3 + \frac{Y}{2}$
ν_L	$1/2$	$1/2$	-1	0
ℓ_L		$-1/2$		-1
ν_R	0	0	0	0
ℓ_R	0	0	-2	-1

	T	T^3	Y	Q
u_L	$1/2$	$1/2$	$1/3$	$2/3$
d_L		$-1/2$		$-1/3$
u_R	0	0	$4/3$	$2/3$
d_R	0	0	$-2/3$	$-1/3$

	T	T^3	Y	Q
ϕ^+	$1/2$	$1/2$	1	1
ϕ^0		$-1/2$		0

$$\left[\frac{\tau^l}{2}, \frac{\tau^m}{2}\right] = i\epsilon_{lmn}\frac{\tau^n}{2} \qquad lmn = 1 \sim 3$$

【Higgs二重項のgauge固定】

$$\Phi(x) \equiv \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \equiv e^{i\left\{\frac{\tau^l}{2}\chi^l(x)\right\}} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \phi(x) \end{pmatrix} \xrightarrow{\text{EWSB}} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \phi(x) \end{pmatrix}$$

$$\frac{v}{\sqrt{2}} \equiv \frac{\mu}{\sqrt{\lambda}} \qquad v\text{: 真空期待値}$$

$$\tilde{\Phi} \equiv i\tau^2\Phi^* \to \frac{1}{\sqrt{2}} \begin{pmatrix} v + \phi(x) \\ 0 \end{pmatrix}$$

【ゲージ場】

$$W_\mu \equiv \frac{1}{\sqrt{2}}(A^1_\mu - i\,A^2_\mu) : W^+ \text{ の場} \qquad W^\dagger_\mu : W^- \text{の場}$$

$$A^1_\mu = \frac{1}{\sqrt{2}}(W_\mu + W^\dagger_\mu) \qquad A^2_\mu = \frac{i}{\sqrt{2}}(W_\mu - W^\dagger_\mu)$$

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \equiv \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A^3_\mu \\ B_\mu \end{pmatrix} \qquad \begin{pmatrix} A^3_\mu \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}$$

$$\sin\theta_W \equiv \frac{g'}{\sqrt{g^2 + g'^2}} \qquad \cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}}$$

$$e \equiv \frac{g\,g'}{\sqrt{g^2 + g'^2}} = g\sin\theta_W = g'\cos\theta_W$$

$$gA^3_\mu = eA_\mu + g\cos\theta_W Z_\mu \qquad g'B_\mu = eA_\mu - \frac{g}{\cos\theta_W}\sin^2\theta_W Z_\mu$$

$$\mathcal{D}_\mu W_\nu \equiv (\partial_\mu + igA^3_\mu)W_\nu = (\partial_\mu + ieA_\mu + ig\cos\theta_W Z_\mu)W_\nu$$

$$F^A_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \qquad F^Z_{\mu\nu} \equiv \partial_\mu Z_\nu - \partial_\nu Z_\mu$$

$$\begin{aligned}
\mathcal{L}_{\text{gauge}} =& -\frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 - \frac{1}{4}(\partial_\mu A^l_\nu - \partial_\nu A^l_\mu - g\epsilon_{lmn}A^m_\mu A^n_\nu)^2 \\
=& -\frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 - \frac{1}{4}(\partial_\mu A^l_\nu - \partial_\nu A^l_\mu)^2 \\
& + \frac{1}{2}g(\partial_\mu A^l_\nu - \partial_\nu A^l_\mu)\epsilon_{lmn}A^{m\mu}A^{n\nu} - \frac{1}{4}(g\epsilon_{lmn}A^m_\mu A^n_\nu)^2 \\
=& -\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} - \frac{1}{4}F^Z_{\mu\nu}F^{Z\mu\nu} - \frac{1}{2}(\partial_\mu W_\nu - \partial_\nu W_\mu)^\dagger(\partial^\mu W^\nu - \partial^\nu W^\mu) \\
& + \frac{1}{2}g(\partial_\mu A^1_\nu - \partial_\nu A^1_\mu)(A^{2\mu}A^{3\nu} - A^{3\mu}A^{2\nu}) \\
& + \frac{1}{2}g(\partial_\mu A^2_\nu - \partial_\nu A^2_\mu)(A^{3\mu}A^{1\nu} - A^{1\mu}A^{3\nu}) \\
& + \frac{1}{2}g(\partial_\mu A^l_\nu - \partial_\nu A^l_\mu)(A^{1\mu}A^{2\nu} - A^{2\mu}A^{1\nu}) \\
& - \frac{g^2}{4}(A^2_\mu A^3_\nu - A^3_\mu A^2_\nu)^2 - \frac{g^2}{4}(A^3_\mu A^1_\nu - A^1_\mu A^3_\nu)^2 - \frac{g^2}{4}(A^1_\mu A^2_\nu - A^2_\mu A^1_\nu)^2 \\
=& -\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} - \frac{1}{4}F^Z_{\mu\nu}F^{Z\mu\nu} - \frac{1}{2}(\partial_\mu W_\nu - \partial_\nu W_\mu)^\dagger(\partial^\mu W^\nu - \partial^\nu W^\mu) \\
& + \frac{i}{4}g\{\partial_\mu(W_\nu + W^\dagger_\nu) - \partial_\nu(W_\mu + W^\dagger_\mu)\}\{(W^\mu - W^{\mu\dagger})A^{3\nu} - A^{3\mu}(W^\nu - W^{\nu\dagger})\} \\
& + \frac{i}{4}g\{\partial_\mu(W_\nu - W^\dagger_\nu) - \partial_\nu(W_\mu - W^\dagger_\mu)\}\{A^{3\mu}(W^\nu + W^{\nu\dagger}) - (W^\mu + W^{\mu\dagger})A^{3\nu}\} \\
& + \frac{i}{4}g(\partial_\mu A^3_\nu - \partial_\nu A^3_\mu)\{(W^\mu + W^{\mu\dagger})(W^\nu - W^{\nu\dagger}) - (W^\mu - W^{\mu\dagger})(W^\nu + W^{\nu\dagger})\} \\
& + \frac{g^2}{8}\{(W_\mu - W^\dagger_\mu)A^3_\nu - A^3_\mu(W_\nu - W^\dagger_\nu)\}^2 \\
& - \frac{g^2}{8}\{A^3_\mu(W_\nu + W^\dagger_\nu) - (W_\mu + W^\dagger_\mu)A^3_\nu\}^2 \\
& + \frac{g^2}{16}\{(W_\mu + W^\dagger_\mu)(W_\nu - W^\dagger_\nu) - (W_\mu - W^\dagger_\mu)(W_\nu + W^\dagger_\nu)\}^2 \\
=& -\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} - \frac{1}{4}F^Z_{\mu\nu}F^{Z\mu\nu} - \frac{1}{2}(\partial_\mu W_\nu - \partial_\nu W_\mu)^\dagger(\partial^\mu W^\nu - \partial^\nu W^\mu) \\
& + \frac{i}{4}g(\partial_\mu W_\nu - \partial_\nu W_\mu + \partial_\mu W^\dagger_\nu - \partial_\nu W^\dagger_\mu)(W^\mu A^{3\nu} - A^{3\mu}W^\nu - W^{\mu\dagger}A^{3\nu} + A^{3\mu}W^{\nu\dagger}) \\
& + \frac{i}{4}g(\partial_\mu W_\nu - \partial_\nu W_\mu - \partial_\mu W^\dagger_\nu + \partial_\nu W^\dagger_\mu)(-W^\mu A^{3\nu} + A^{3\mu}W^\nu - W^{\mu\dagger}A^{3\nu} + A^{3\mu}W^{\nu\dagger}) \\
& + \frac{i}{2}g(\partial_\mu A^3_\nu - \partial_\nu A^3_\mu)(W^{\mu\dagger}W^\nu - W^{\nu\dagger}W^\mu) \\
& + \frac{g^2}{8}(W_\mu A^3_\nu - W^\dagger_\mu A^3_\nu - A^3_\mu W_\nu + A^3_\mu W^\dagger_\nu)^2 \\
& - \frac{g^2}{8}(A^3_\mu W_\nu + A^3_\mu W^\dagger_\nu - W_\mu A^3_\nu - W^\dagger_\mu A^3_\nu)^2 \\
& + \frac{g^2}{4}(W^\dagger_\mu W_\nu - W_\mu W^\dagger_\nu)^2 \\
=& -\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} - \frac{1}{4}F^Z_{\mu\nu}F^{Z\mu\nu} - \frac{1}{2}(\partial_\mu W_\nu - \partial_\nu W_\mu)^\dagger(\partial^\mu W^\nu - \partial^\nu W^\mu) \\
& - \frac{i}{2}g\{(\partial_\mu W_\nu - \partial_\nu W_\mu)(W^{\mu\dagger}A^{3\nu} - A^{3\mu}W^{\nu\dagger}) - (\partial_\mu W^\dagger_\nu - \partial_\nu W^\dagger_\mu)(W^\mu A^{3\nu} - A^{3\mu}W^\nu)\} \\
& + \frac{i}{2}g(\partial_\mu A^3_\nu - \partial_\nu A^3_\mu)(W^{\mu\dagger}W^\nu - W^{\nu\dagger}W^\mu)
\end{aligned}$$

$$-\frac{g^2}{2}(W_\mu A^3_\nu - A^3_\mu W_\nu)(W^{\mu\dagger}A^{3\nu} - A^{3\mu}W^{\nu\dagger})$$

$$+\frac{g^2}{2}(W^\dagger_\mu W^{\dagger\mu} W_\nu W^\nu - W_\mu W^\nu W^{\dagger\mu} W^\dagger_\nu)$$

$$= -\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} - \frac{1}{4}F^Z_{\mu\nu}F^{Z\mu\nu}$$

$$-\frac{1}{2}\{\partial_\mu W_\nu - \partial_\nu W_\mu - ig(W_\mu A^3_\nu - A^3_\mu W_\nu)\}^\dagger\{\partial^\mu W^\nu - \partial^\nu W^\mu - ig(W^\mu A^{3\nu} - A^{3\mu}W^\nu)\}$$

$$+ig(\partial_\mu A^3_\nu - \partial_\nu A^3_\mu)W^{\mu\dagger}W^\nu$$

$$+\frac{g^2}{2}(|W_\mu W^\mu|^2 - |W_\mu W^\nu|^2) - \frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu}$$

$$= -\frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} - \frac{1}{4}F^Z_{\mu\nu}F^{Z\mu\nu} - \frac{1}{2}(\mathcal{D}_\mu W_\nu - \mathcal{D}_\nu W_\mu)^\dagger(\mathcal{D}^\mu W^\nu - \mathcal{D}^\nu W^\mu)$$

$$+i(eF^A_{\mu\nu} + g\cos\theta_W F^Z_{\mu\nu})W^{\mu\dagger}W^\nu + \frac{g^2}{2}(|W_\mu W^\mu|^2 - |W_\mu W^\nu|^2)$$

【Higgs場】

$$V(\Phi) \equiv -\mu^2\Phi^\dagger\Phi + \frac{\lambda}{2}(\Phi^\dagger\Phi)^2 = -\frac{1}{2}\mu^2(v+\phi)^2 + \frac{\lambda}{8}(v+\phi)^4$$

$$= -\frac{1}{2}\mu^2(v^2 + 2v\phi + \phi^2) + \frac{\lambda}{8}(v^4 + 4v^3\phi + 6v^2\phi^2 + 4v\phi^3 + \phi^4)$$

$$= -\frac{1}{2}\mu^2 v^2 + \frac{\lambda}{8}v^4 + \left(-\mu^2 v + \frac{\lambda}{2}v^3\right)\phi + \left(-\frac{1}{2}\mu^2 + \frac{3\lambda}{4}v^2\right)\phi^2 + \frac{\lambda}{2}v\phi^3 + \frac{\lambda}{8}\phi^4$$

$$V(v) \equiv -\frac{1}{2}\mu^2 v^2 + \frac{\lambda}{8}v^4 \qquad V'(v) = -\mu^2 v + \frac{\lambda}{2}v^3 = 0 \to \frac{v^2}{2} = \frac{\mu^2}{\lambda}$$

$$M_W \equiv \frac{gv}{2} \qquad M_Z \equiv \frac{1}{2}\sqrt{g^2 + g'^2}\,v = \frac{M_W}{\cos\theta_W} \qquad m \equiv \sqrt{2}\mu = v\sqrt{\lambda}：\text{Higgs質量}$$

$$\tau^l A^l_\mu = \begin{pmatrix} A^3_\mu & \sqrt{2}W_\mu \\ \sqrt{2}W^\dagger_\mu & -A^3_\mu \end{pmatrix}$$

$$\mathcal{L}_{\text{Higgs}} = \left|\left(i\partial_\mu - g'\frac{1}{2}Y_H B_\mu - g\frac{\tau^l}{2}A^l_\mu\right)\Phi\right|^2 + \mu^2\Phi^\dagger\Phi - \frac{\lambda}{2}(\Phi^\dagger\Phi)^2$$

$$= \frac{1}{2}\left|\begin{pmatrix} i\partial_\mu - g'\frac{1}{2}B_\mu - \frac{g}{2}A^3_\mu & -\frac{g}{\sqrt{2}}W_\mu \\ -\frac{g}{\sqrt{2}}W^\dagger_\mu & i\partial_\mu - g'\frac{1}{2}B_\mu + \frac{g}{2}A^3_\mu \end{pmatrix}\begin{pmatrix} 0 \\ v+\phi \end{pmatrix}\right|^2 + \mu^2\Phi^\dagger\Phi - \frac{\lambda}{2}(\Phi^\dagger\Phi)^2$$

$$= \frac{1}{2}\left|\begin{pmatrix} -\frac{g}{\sqrt{2}}W_\mu(v+\phi) \\ \left(i\partial_\mu - \frac{g'}{2}B_\mu + \frac{g}{2}A^3_\mu\right)(v+\phi) \end{pmatrix}\right|^2 + \frac{1}{2}\mu^2(v+\phi)^2 - \frac{\lambda}{8}(v+\phi)^4$$

$$= \frac{g^2}{4}W^\dagger_\mu W^\mu(v+\phi)^2 + \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{8}|-g'B_\mu + gA^3_\mu|^2(v+\phi)^2$$

$$+\mu^2\left(\frac{v}{\sqrt{2}}\right)^2 - \frac{\lambda}{2}\left(\frac{v}{\sqrt{2}}\right)^4 + v\lambda\left(\frac{\mu^2}{\lambda} - \frac{v^2}{2}\right)\phi + \left(\frac{\mu^2}{2} - \frac{3\lambda v^2}{4}\right)\phi^2 - \frac{\lambda v}{2}\phi^3 - \frac{\lambda}{8}\phi^4$$

$$= \frac{g^2}{4}W^\dagger_\mu W^\mu(v+\phi)^2 + \frac{1}{8}\frac{g^2}{\cos^2\theta_W}Z^2_\mu(v+\phi)^2$$

$$+\frac{1}{2}(\partial_\mu\phi)^2-\frac{\lambda v^2}{2}\phi^2-\frac{\lambda v}{2}\phi^3-\frac{\lambda}{8}\phi^4-V\left(\frac{v}{\sqrt{2}}\right)$$

$$=\frac{g^2v^2}{4}W_\mu^\dagger W^\mu+\frac{g^2v^2}{8\cos^2\theta_W}Z_\mu^2+\frac{g^2(2v\phi+\phi^2)}{4}\left(W_\mu^\dagger W^\mu+\frac{1}{2\cos^2\theta_W}Z_\mu^2\right)$$

$$+\frac{1}{2}(\partial_\mu\phi)^2-\frac{\lambda v^2}{2}\phi^2-\frac{\lambda v}{2}\phi^3-\frac{\lambda}{8}\phi^4-V\left(\frac{v}{\sqrt{2}}\right)$$

$$=M_W^2W_\mu^\dagger W^\mu+\frac{1}{2}M_Z^2Z_\mu^2+\left(gM_W\phi+\frac{g^2}{4}\phi^2\right)\left(W_\mu^\dagger W^\mu+\frac{1}{2\cos^2\theta_W}Z_\mu^2\right)$$

$$+\frac{1}{2}[(\partial_\mu\phi)^2-m^2\phi^2]-\frac{m\sqrt{\lambda}}{2}\phi^3-\frac{\lambda}{8}\phi^4-V\left(\frac{v}{\sqrt{2}}\right)$$

【Lepton運動項】

$J_\ell^\mu\equiv\bar{\ell}_L^j\gamma^\mu\nu_L^j$: Weak charged current (Pure left-handed)

$J_{\ell,\mathrm{EM}}^\mu\equiv(-)\bar{\ell}^j\gamma^\mu\ell^j=-\bar{\ell}_L^j\gamma^\mu\ell_L^j-\bar{\ell}_R^j\gamma^\mu\ell_R^j$: EM current

$\ell^j\equiv\ell_L^j+\ell_R^j \qquad \nu^j\equiv\nu_L^j+\nu_R^j$

$\bar{\ell}_L^j\gamma^\mu\ell_R^j=\bar{\ell}_R^j\gamma^\mu\ell_L^j=\bar{\ell}_L^j\ell_L^j=\bar{\ell}_R^j\ell_R^j=0$ に注意

$J_\ell^{3\mu}\equiv\bar{L}_L^j\gamma^\mu\frac{\tau^3}{2}L_L^j=\frac{1}{2}\bar{\nu}_L^j\gamma^\mu\nu_L^j-\frac{1}{2}\bar{\ell}_L^j\gamma^\mu\ell_L^j$

$J_{\ell,Z}^\mu\equiv J_\ell^{3\mu}-\sin^2\theta_W J_{\ell,\mathrm{EM}}^\mu$: Weak neutral current (Pure left-handed ではない)

$$\mathcal{L}_{\mathrm{lepton}}^{\mathrm{kin}}=\bar{L}_L^j\gamma^\mu\left[i\partial_\mu-g'\frac{Y_{\ell L}}{2}B_\mu-g\frac{\tau^l}{2}A_\mu^l\right]L_L^j+\bar{\ell}_R^j\gamma^\mu\left[i\partial_\mu-g'\frac{Y_{\ell R}}{2}B_\mu\right]\ell_R^j+\bar{\nu}_R^j\gamma^\mu i\partial_\mu\nu_R^j$$

L_L^j やℓ_R^j, ν_R^j は世代間で任意のユニタリー回転しても上式は不変

$$=\bar{\ell}^j i\partial\!\!\!/\ell^j+\bar{\nu}^j i\partial\!\!\!/\nu^j+\frac{1}{2}\bar{L}_L^j\gamma^\mu\begin{pmatrix} g'B_\mu-gA_\mu^3 & -\sqrt{2}gW_\mu \\ -\sqrt{2}gW_\mu^\dagger & g'B_\mu+gA_\mu^3\end{pmatrix}L_L^j+g'B_\mu\bar{\ell}_R^j\gamma^\mu\ell_R^j$$

$$=\bar{\ell}^j i\partial\!\!\!/\ell^j+\bar{\nu}^j i\partial\!\!\!/\nu^j+g'B_\mu\bar{\ell}_R^j\gamma^\mu\ell_R^j$$

$$+\frac{1}{2}\bar{L}_L^j\gamma^\mu\begin{pmatrix} -\frac{g}{\cos\theta_W}Z_\mu & -\sqrt{2}gW_\mu \\ -\sqrt{2}gW_\mu^\dagger & 2eA_\mu+\frac{g}{\cos\theta_W}(1-2\sin^2\theta_W)Z_\mu\end{pmatrix}L_L^j$$

$$=\bar{\ell}^j i\partial\!\!\!/\ell^j+\bar{\nu}^j i\partial\!\!\!/\nu^j+\left(eA_\mu-\frac{g}{\cos\theta_W}\sin^2\theta_W Z_\mu\right)\bar{\ell}_R^j\gamma^\mu\ell_R^j-\frac{g}{\sqrt{2}}(\bar{\nu}_L^j\gamma^\mu\ell_L^jW_\mu+\bar{\ell}_L^j\gamma^\mu\nu_L^jW_\mu^\dagger)$$

$$-\frac{g}{2\cos\theta_W}\bar{\nu}_L^j\gamma^\mu\nu_L^jZ_\mu+\left\{eA_\mu+\frac{g}{2\cos\theta_W}(1-2\sin^2\theta_W)Z_\mu\right\}\bar{\ell}_L^j\gamma^\mu\ell_L^j$$

$$=\bar{\ell}^j i\partial\!\!\!/\ell^j+\bar{\nu}^j i\partial\!\!\!/\nu^j-\frac{g}{\sqrt{2}}\{(\bar{\ell}_L^j\gamma^\mu\nu_L^j)^\dagger W_\mu+\bar{\ell}_L^j\gamma^\mu\nu_L^jW_\mu^\dagger\}$$

$$+e\bar{\ell}^j\gamma^\mu\ell^jA_\mu-\frac{g}{\cos\theta_W}\left(\frac{1}{2}\bar{\nu}_L^j\gamma^\mu\nu_L^j-\frac{1}{2}\bar{\ell}_L^j\gamma^\mu\ell_L^j+\sin^2\theta_W\bar{\ell}^j\gamma^\mu\ell^j\right)Z_\mu$$

$$=\bar{\ell}^j i\partial\!\!\!/\ell^j+\bar{\nu}^j i\partial\!\!\!/\nu^j-\frac{g}{\sqrt{2}}(J_\ell^{\mu\dagger}W_\mu+J_\ell^\mu W_\mu^\dagger)-eJ_{\ell,\mathrm{EM}}^\mu A_\mu-\frac{g}{\cos\theta_W}J_{\ell,Z}^\mu Z_\mu$$

【Lepton質量項】

$V_{\ell L}\, Y_\ell V_{\ell R}^\dagger \equiv \mathrm{diag}(y_\ell^j) \qquad V_{\nu L}\, Y_\nu V_{\nu R}^\dagger \equiv \mathrm{diag}(y_\nu^j)$

bi-unitary変換で非負実数固有値の対角行列化が可能 $Y_D = V_L Y V_R^\dagger$

$y_\ell^j \geqslant 0 \qquad y_\nu^j \geqslant 0$

$V_{\ell L} V_{\nu L}^\dagger \equiv U_\nu$ (U_ν: PMNS matrix)

$L_L^j \equiv V_{\ell L}^{jk} L'^k_L \qquad \ell_R^j \equiv V_{\ell R}^{jk} \ell'^k_R \qquad \nu_R^j \equiv (U_\nu V_{\nu R})^{jk} \nu'^k_R$

$L'^j_L = V_{\ell L}^{\dagger jk} L_L^k \qquad \ell'^j_R = V_{\ell R}^{\dagger jk} \ell_R^k \qquad \nu'^j_R = (U_\nu V_{\nu R})^{\dagger jk} \nu_R^k = (V_{\nu R}^\dagger U_\nu^\dagger)^{jk} \nu_R^k$

prime(')の付いた場は,「再定義の前に」便宜上定義されていた世代での場

$\hat{\nu}^j \equiv U_\nu^{\dagger jk} (\nu_L + \nu_R)^k = U_\nu^{\dagger jk} \nu^k$: 質量固有状態の neutrino

$m_\ell^j \equiv \dfrac{y_\ell^j v}{\sqrt{2}} \qquad m_\nu^j \equiv \dfrac{y_\nu^j v}{\sqrt{2}}$

$$\begin{aligned}
\mathcal{L}_{\text{lepton}}^{\text{mass}} &= -(\bar{L}'^j_L \Phi) Y_\ell^{jk} \ell'^k_R - (\bar{L}'^j_L \tilde{\Phi}) Y_\nu^{jk} \nu'^k_R + h.c. \\
&= -(\bar{L}_L^j \Phi)(V_{\ell L}\, Y_\ell V_{\ell R}^\dagger)^{jk} \ell_R^k - (\bar{L}_L^j \tilde{\Phi})(V_{\ell L} Y_\nu V_{\nu R}^\dagger U_\nu^\dagger)^{jk} \nu_R^k + h.c. \\
&= -(\bar{L}_L^j \Phi)(V_{\ell L}\, Y_\ell V_{\ell R}^\dagger)^{jk} \ell_R^k - (\bar{L}_L^j \tilde{\Phi})(V_{\ell L} V_{\nu L}^\dagger V_{\nu L} Y_\nu V_{\nu R}^\dagger U_\nu^\dagger)^{jk} \nu_R^k + h.c. \\
&= -y_\ell^j (\bar{L}_L^j \Phi) \ell_R^j - y_\nu^j [(\bar{L}_L U_\nu)^j \tilde{\Phi}] (U_\nu^\dagger \nu_R)^j + h.c. \\
&= -\frac{1}{\sqrt{2}} y_\ell^j \bar{\ell}_L^j (v+\phi) \ell_R^j - \frac{1}{\sqrt{2}} y_\nu^j (\bar{\nu}_L U_\nu)^j (v+\phi)(U_\nu^\dagger \nu_R)^j + h.c. \\
&= -\frac{1}{\sqrt{2}} y_\ell^j \bar{\ell}_L^j (v+\phi) \ell_R^j - \frac{1}{\sqrt{2}} y_\nu^j \bar{\hat{\nu}}_L^j (v+\phi) \hat{\nu}_R^j + h.c.
\end{aligned}$$

$$(\bar{\ell}_L \ell_R)^\dagger = (\ell_L^\dagger \gamma^0 \ell_R)^\dagger = \ell_R^\dagger \gamma^0 \ell_L = \bar{\ell}_R \ell_L$$

$$\bar{\ell}_L \ell_R + (\bar{\ell}_L \ell_R)^\dagger = \bar{\ell}_L \ell_R + \bar{\ell}_R \ell_L = (\bar{\ell}_L + \bar{\ell}_R)(\ell_L + \ell_R) = \bar{\ell}\ell$$

$$= -\left(m_\ell^j + \frac{y_\ell^j}{\sqrt{2}} \phi \right) \bar{\ell}^j \ell^j - \left(m_\nu^j + \frac{y_\nu^j}{\sqrt{2}} \phi \right) \bar{\hat{\nu}}^j \hat{\nu}^j$$

$\Phi(x) \equiv \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \to \dfrac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+\phi(x) \end{pmatrix}$ によって, $\mathrm{SU}(2)_L$回転の空間に第3軸が定義され, $L = \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix}$ でも第1, 2軸周りの回転 (2重項の上下成分を混合する操作) は, もうできない (元々 我々 は上成分を ν_L, 下成分を ℓ_L と定義した) ことになるが, 第3軸周りの回転 $u_3(\varphi) = \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}$ は, まだ自由にできる。また, Lには元々の$U(1)$回転 $e^{i\alpha}$ (2重項の上下成分同時回転)の自由度もあるので, 両方合わせると, ν_L, ℓ_L 各々の$U(1)$回転の自由度とすることができる。

【Quark運動項】

$J_q^\mu \equiv \bar{d}_L^j \gamma^\mu u_L^j$: Weak charged current (Pure left-handed)

$J_{q,\mathrm{EM}}^\mu \equiv \left(\dfrac{2}{3}\right) \bar{u}^j \gamma^\mu u^j + \left(-\dfrac{1}{3}\right) \bar{d}^j \gamma^\mu d^j$: EM current

$u^j \equiv u_L^j + u_R^j \qquad d^j \equiv d_L^j + d_R^j$

$J_q^{3\mu} \equiv \bar{Q}_L^j \gamma^\mu \dfrac{\tau^3}{2} Q_L^j = \dfrac{1}{2} \bar{u}_L^j \gamma^\mu u_L^j - \dfrac{1}{2} \bar{d}_L^j \gamma^\mu d_L^j$

$J^{\mu}_{q,Z} \equiv J^{3\mu}_{q} - \sin^2\theta_W J^{\mu}_{q,\mathrm{EM}}$: Weak neutral current (Pure left-handed ではない)

$$\mathcal{L}^{\mathrm{kin}}_{\mathrm{quark}} = \bar{Q}^j_L \gamma^\mu \left[i\partial_\mu - g' \frac{Y_{QL}}{2} B_\mu - g \frac{\tau^l}{2} A^l_\mu \right] Q^j_L$$

$$+ \bar{u}^j_R \gamma^\mu \left[i\partial_\mu - g' \frac{Y_{uR}}{2} B_\mu \right] u^j_R$$

$$+ \bar{d}^j_R \gamma^\mu \left[i\partial_\mu - g' \frac{Y_{dR}}{2} B_\mu \right] d^j_R$$

$$= \bar{u}^j i \partial\!\!\!/ u^j + \bar{d}^j i \partial\!\!\!/ d^j + \frac{1}{2} \bar{Q}^j_L \gamma^\mu \begin{pmatrix} -\frac{1}{3} g' B_\mu - g A^3_\mu & -\sqrt{2} g W_\mu \\ -\sqrt{2} g W^\dagger_\mu & -\frac{1}{3} g' B_\mu + g A^3_\mu \end{pmatrix} Q^j_L$$

$$- \frac{2}{3} g' B_\mu \bar{u}^j_R \gamma^\mu u^j_R + \frac{1}{3} g' B_\mu \bar{d}^j_R \gamma^\mu d^j_R$$

$$= \bar{u}^j i \partial\!\!\!/ u^j + \bar{d}^j i \partial\!\!\!/ d^j$$

$$+ \bar{Q}^j_L \gamma^\mu \begin{pmatrix} -\frac{2}{3} e A_\mu + \frac{g\left(\frac{4}{3}\sin^2\theta_W - 1\right)}{2\cos\theta_W} Z_\mu & -\frac{g}{\sqrt{2}} W_\mu \\ -\frac{g}{\sqrt{2}} W^\dagger_\mu & \frac{1}{3} e A_\mu + \frac{g\left(-\frac{2}{3}\sin^2\theta_W + 1\right)}{2\cos\theta_W} Z_\mu \end{pmatrix} Q^j_L$$

$$- \frac{2}{3} \left(e A_\mu - \frac{g}{\cos\theta_W} \sin^2\theta_W Z_\mu \right) \bar{u}^j_R \gamma^\mu u^j_R + \frac{1}{3} \left(e A_\mu - \frac{g}{\cos\theta_W} \sin^2\theta_W Z_\mu \right) \bar{d}^j_R \gamma^\mu d^j_R$$

$$= \bar{u}^j i \partial\!\!\!/ u^j + \bar{d}^j i \partial\!\!\!/ d^j - \frac{g}{\sqrt{2}} \bar{u}^j_L \gamma^\mu d^j_L W_\mu - \frac{g}{\sqrt{2}} \bar{d}^j_L \gamma^\mu u^j_L W^\dagger_\mu - \frac{2}{3} e \bar{u}^j \gamma^\mu u^j A_\mu + \frac{1}{3} e \bar{d}^j \gamma^\mu d^j A_\mu$$

$$- \frac{g}{\cos\theta_W} \left\{ \left(\frac{1}{2} - \frac{2}{3} \sin^2\theta_W \right) \bar{u}^j_L \gamma^\mu u^j_L - \frac{2}{3} \sin^2\theta_W \bar{u}^j_R \gamma^\mu u^j_R \right\} Z_\mu$$

$$- \frac{g}{\cos\theta_W} \left\{ \left(\frac{1}{3} \sin^2\theta_W - \frac{1}{2} \right) \bar{d}^j_L \gamma^\mu d^j_L + \frac{1}{3} \sin^2\theta_W \bar{d}^j_R \gamma^\mu d^j_R \right\} Z_\mu$$

$$= \bar{u}^j i \partial\!\!\!/ u^j + \bar{d}^j i \partial\!\!\!/ d^j - \frac{g}{\sqrt{2}} (J^{\mu\dagger}_q W_\mu + J^\mu_q W^\dagger_\mu) - e J^\mu_{q,\mathrm{EM}} A_\mu - \frac{g}{\cos\theta_W} J^\mu_{q,Z} Z_\mu$$

【Quark質量項】

$V_{uL} Y_u V^\dagger_{uR} \equiv \mathrm{diag}(y^j_u)$ $\quad$ $V_{dL} Y_d V^\dagger_{dR} \equiv \mathrm{diag}(y^j_d)$

$V_{uL} V^\dagger_{dL} \equiv U_d$ (U_d: CKM matrix)

$Q^j_L \equiv V^{jk}_{uL} Q'^k_L$ $\quad$ $u^j_R \equiv V^{jk}_{uR} u'^k_R$ $\quad$ $d^j_R \equiv (U_d V_{dR})^{jk} d'^k_R$

$Q'^j_L = V^{\dagger jk}_{uL} Q^k_L$ $\quad$ $u'^j_R = V^{\dagger jk}_{uR} u^k_R$ $\quad$ $d'^j_R = (V^\dagger_{dR} U^\dagger_d)^{jk} d^k_R$

prime(′)の付いた場は,「再定義の前に」便宜上定義されていた世代での場

$\hat{d}^j \equiv U^{\dagger jk}_d (d_L + d_R)^k = U^{\dagger jk}_d d^k$: 質量固有状態の down-type quark

$m^j_d \equiv \frac{y^j_d v}{\sqrt{2}}$ $\quad$ $m^j_u \equiv \frac{y^j_u v}{\sqrt{2}}$

$$\mathcal{L}^{\mathrm{mass}}_{\mathrm{quark}} = -(\bar{Q}'^j_L \Phi) Y^{jk}_d d'^k_R - (\bar{Q}'^j_L \tilde{\Phi}) Y^{jk}_u u'^k_R + h.c.$$

$$=-(\bar{Q}_L^j\Phi)(V_{uL}Y_dV_{dR}^\dagger U_d^\dagger)^{jk}d_R^k-(\bar{Q}_L^j\tilde{\Phi})(V_{uL}Y_uV_{uR}^\dagger)^{jk}u_R^k+h.c.$$

$$=-y_d^j[(\bar{Q}_LV_{uL}V_{dL}^\dagger)^j\Phi](U_d^\dagger d_R)^j-y_u^j(\bar{Q}_L^j\tilde{\Phi})u_R^j+h.c.$$

$$=-y_d^j[(\bar{Q}_LU_d)^j\Phi](U_d^\dagger d_R)^j-y_u^j(\bar{Q}_L^j\tilde{\Phi})u_R^j+h.c.$$

$$=-\frac{1}{\sqrt{2}}y_d^j\bar{\hat{d}}_L^j(v+\phi)\hat{d}_R^j-\frac{1}{\sqrt{2}}y_u^j\bar{u}_L^j(v+\phi)u_R^j+h.c.$$

$$=-\left(m_d^j+\frac{y_d^j}{\sqrt{2}}\phi\right)\bar{\hat{d}}^j\hat{d}^j-\left(m_u^j+\frac{y_u^j}{\sqrt{2}}\phi\right)\bar{u}^ju^j$$

「定理: bi-unitary 変換で正の実数固有値の対角行列化が可能」の証明

任意の $n\times n$ complex matrix Mに対して,

$N\equiv M^\dagger M$　　　N は Hermitian

を定義する。任意の非零複素ベクトル $z(\in C^n, z\neq 0)$ に対して

$z^\dagger Nz=z^\dagger M^\dagger Mz=|Mz|^2\geqslant 0$

となる。すなわち行列 N は半正定値行列 (positive semidefinite)である。

半正定値行列は, 非負の実数固有値を持つ。

なぜなら x をN の固有ベクトル($x^\dagger x=1$), 固有値をλ とすると,

$Nx=\lambda x$

となるが, N は半正定値行列なので, $\lambda=x^\dagger Nx\geqslant 0$

よって, Nは, 適当なユニタリ行列で

$V^\dagger NV=V^\dagger M^\dagger MV=\mathrm{diag}(\lambda_i)$　　　$\lambda_i\geqslant 0, \in R$

と対角化出来る。

$M_D\equiv\mathrm{diag}\left(\frac{1}{\sqrt{\lambda_i}}\right)$ という行列を定義すると, $M_D^\dagger=M_D$

$U\equiv MVM_D$ は, ユニタリ行列。なぜなら

$U^\dagger U=M_DV^\dagger M^\dagger MVM_D=M_D\mathrm{diag}(\lambda_i)M_D=\mathrm{diag}\left(\frac{1}{\sqrt{\lambda_i}}\right)\mathrm{diag}(\lambda_i)\mathrm{diag}\left(\frac{1}{\sqrt{\lambda_i}}\right)=1$

したがって

$U^\dagger MV=M_DV^\dagger M^\dagger MV=M_D\mathrm{diag}(\lambda_i)=\mathrm{diag}(\sqrt{\lambda_i})$

と非負成分で対角化できた。
