Normalization of Quadratic Curve

General quadratic curve is given by

$$f(x,y) = a\,x^2 + b\,y^2 + c\,x\,y + d\,x + e\,y + f = 0 \tag{1}$$

We define $\boldsymbol{x}$, $\boldsymbol{A}$, and $\boldsymbol{B}$ by

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{2}$$

$$\boldsymbol{A} = \begin{pmatrix} a & c/2 \\ c/2 & b \end{pmatrix} \tag{3}$$

$$\boldsymbol{B} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{4}$$

Then Eq.(1) is rewritten like

$$f(x,y) = {}^t\boldsymbol{x}\,\boldsymbol{A}\,\boldsymbol{x} + {}^t\boldsymbol{B}\,\boldsymbol{x} + f \tag{5}$$

At first, we attempt parallel translation so that the center of the curve is at origin. We define the center by $\boldsymbol{x}_0$ like:

$$\boldsymbol{x} = \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \boldsymbol{x}' + \boldsymbol{x}_0 \tag{6}$$

and Eq.(1) is now

$$\begin{aligned} f(x,y) &= {}^t(\boldsymbol{x}' + \boldsymbol{x}_0)\,\boldsymbol{A}\,(\boldsymbol{x}' + \boldsymbol{x}_0) + {}^t\boldsymbol{B}\,(\boldsymbol{x}' + \boldsymbol{x}_0) + f \\ &= {}^t\boldsymbol{x}'\,\boldsymbol{A}\,\boldsymbol{x}' + 2\,{}^t\boldsymbol{x}_0\,\boldsymbol{A}\,\boldsymbol{x}' + {}^t\boldsymbol{B}\,\boldsymbol{x}' + {}^t\boldsymbol{x}_0\,\boldsymbol{A}\,\boldsymbol{x}_0 + {}^t\boldsymbol{B}\,\boldsymbol{x}_0 + f \\ &= {}^t\boldsymbol{x}'\,\boldsymbol{A}\,\boldsymbol{x}' + f' \end{aligned} \tag{7}$$

where

$$f' = {}^t\boldsymbol{x}_0\,\boldsymbol{A}\,\boldsymbol{x}_0 + {}^t\boldsymbol{B}\,\boldsymbol{x}_0 + f \tag{8}$$

and we assume the following condition so that after translation opration the curve is placed at origin in its center.

$$2\,\boldsymbol{A}\,\boldsymbol{x}_0 + \boldsymbol{B} = 0 \tag{9}$$

Same equation as Eq.(9) is derived as well by

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial y} f(x,y) = 0 \tag{10}$$

Now we get

$$\boldsymbol{x}_0 = -\frac{1}{2} \cdot \boldsymbol{A}^{-1}\boldsymbol{B} \tag{11}$$

$$f' = -\frac{1}{4}\,{}^t\boldsymbol{B}\,\boldsymbol{A}^{-1}\,\boldsymbol{B} + f \tag{12}$$

As the next step, we rotate the axis.

$$\tilde{\boldsymbol{x}} = U^{-1}\boldsymbol{x}' \tag{13}$$

$$f(x,y) = {}^t\boldsymbol{x}'\,\boldsymbol{A}\,\boldsymbol{x}' + f' = {}^t\tilde{\boldsymbol{x}}\,{}^t\boldsymbol{U}\boldsymbol{A}\boldsymbol{U}\,\tilde{\boldsymbol{x}} + f' \tag{14}$$

where, ${}^t\boldsymbol{U}\boldsymbol{A}\boldsymbol{U}$ supposed to be diagonal matrix.