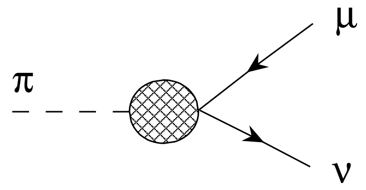
Y. Takeuchi, 2008.8

Helicity of Atmospheric Muon

Muon Helicity in Pion decay

Considering $\pi^+ \to \mu^+\nu$



In the diagram above, μ^+ is incoming, while ν outgoin $\Rightarrow\ \bar{u}\,(\nu)\gamma^\mu\frac{1-\gamma^5}{2}v(\mu^+)$,

k, p, q: 4 momentum of π, μ, and ν, respectively, where $k=p+q$

u, v: Dirac spinor of ν and μ^+, respectively

$$\begin{aligned}\mathfrak{M} &= f_\pi k_\mu\cdot\bar{u}\gamma^\mu\frac{1-\gamma^5}{2}v = f_\pi(p_\mu+q_\mu)\bar{u}\gamma^\mu\frac{1-\gamma^5}{2}v \\ &= f_\pi\bar{u}\,(\not{p}+\not{q})\frac{1-\gamma^5}{2}v \\ &= f_\pi\bar{u}\,\frac{1+\gamma^5}{2}\not{p}v = -f_\pi m_\mu\bar{u}\,\frac{1+\gamma^5}{2}v\end{aligned},$$

where f_π is a form factor of pion. The following relations are used in the calculation above:

$$(\not{p}+m_\mu)v=0 \qquad \bar{u}\,(\not{q}-m_\nu)=0 \qquad \gamma^\mu(1-\gamma^5)=(1+\gamma^5)\gamma^\mu$$

Describe u and v with 2 component spinor ϕ and χ as

$$u=\sqrt{E_\nu}\begin{pmatrix}\phi\\ \frac{\vec{\sigma}\cdot\vec{q}}{E_\nu}\phi\end{pmatrix}=\sqrt{E_\nu}\begin{pmatrix}\phi\\ \vec{\sigma}\cdot\vec{\beta}\phi\end{pmatrix}$$

$$v=\sqrt{E_\mu+m_\mu}\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E_\mu+m_\mu}\chi\\ \chi\end{pmatrix}=\sqrt{E_\mu+m_\mu}\begin{pmatrix}\vec{\sigma}\cdot\vec{\alpha}\chi\\ \chi\end{pmatrix},$$

where $\vec{\beta}\equiv\frac{\vec{q}}{|\vec{q}\,|}$, $\vec{\alpha}\equiv\frac{\vec{p}}{E_\mu+m_\mu}$. Then,

$$\begin{aligned}\mathfrak{M} &= -\frac{f_\pi m_\mu}{2}\bar{u}\,(1+\gamma^5)v \\ &= -\frac{f_\pi m_\mu}{2}\sqrt{E_\nu}\sqrt{E_\mu+m_\mu}\begin{pmatrix}\phi^\dagger & \phi^\dagger\vec{\sigma}\cdot\vec{\beta}\end{pmatrix}\gamma^0(1+\gamma^5)\begin{pmatrix}\vec{\sigma}\cdot\vec{\alpha}\chi\\ \chi\end{pmatrix} \\ &= -\frac{f_\pi m_\mu}{2}\sqrt{E_\nu}\sqrt{E_\mu+m_\mu}\begin{pmatrix}\phi^\dagger & -\phi^\dagger\vec{\sigma}\cdot\vec{\beta}\end{pmatrix}\begin{pmatrix}(1+\vec{\sigma}\cdot\vec{\alpha})\chi\\ (1+\vec{\sigma}\cdot\vec{\alpha})\chi\end{pmatrix} \\ &= -\frac{f_\pi m_\mu}{2}\sqrt{E_\nu}\sqrt{E_\mu+m_\mu}\left\{\phi^\dagger(1+\vec{\sigma}\cdot\vec{\alpha})\chi-\phi^\dagger\vec{\sigma}\cdot\vec{\beta}\,(1+\vec{\sigma}\cdot\vec{\alpha})\chi\right\} \\ &= -\frac{f_\pi m_\mu}{2}\sqrt{E_\nu}\sqrt{E_\mu+m_\mu}\phi^\dagger(1-\vec{\sigma}\cdot\vec{\beta})(1+\vec{\sigma}\cdot\vec{\alpha})\chi\end{aligned}$$

(To check the calculation) Suppose pion rest frame case($\vec{\beta}=\hat{z}$, $\vec{\alpha}=-\alpha\hat{z}$)

$$k^\mu=(m_\pi;\vec{0})$$

$$\alpha = \frac{p^*}{E_\mu^* + m_\mu} = \frac{\frac{m_\pi^2 - m_\mu^2}{2m_\pi}}{\frac{m_\pi^2 + m_\mu^2}{2m_\pi} + m_\mu} = \frac{m_\pi^2 - m_\mu^2}{(m_\pi + m_\mu)^2} = \frac{m_\pi - m_\mu}{m_\pi + m_\mu}$$

$$\begin{aligned}
k_\mu \cdot \bar{u}\gamma^\mu(1-\gamma^5)v &= -m_\pi\sqrt{E_\nu^*}\sqrt{E_\mu^* + m_\mu}\,\phi^\dagger(1 - \vec{\sigma}\cdot\vec{\beta})(1 - \vec{\sigma}\cdot\vec{\alpha})\chi \\
&= -m_\pi\sqrt{\frac{m_\pi^2 - m_\mu^2}{2m_\pi}}\sqrt{\frac{m_\pi^2 + m_\mu^2}{2m_\pi} + m_\mu}\,\phi^\dagger(1-\sigma_3)(1+\alpha\sigma_3)\chi \\
&= -\frac{m_\pi + m_\mu}{2}\sqrt{m_\pi^2 - m_\mu^2}\,\phi^\dagger(1-\sigma_3)(1+\alpha\sigma_3)\chi \\
&= -\frac{m_\pi + m_\mu}{2}\sqrt{m_\pi^2 - m_\mu^2}\,\phi^\dagger\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1+\alpha & 0 \\ 0 & 1-\alpha \end{pmatrix}\chi \\
&= -(m_\pi + m_\mu)\sqrt{m_\pi^2 - m_\mu^2}\begin{pmatrix} 0 & 1-\alpha \end{pmatrix}\chi \\
&= -(m_\pi + m_\mu)\sqrt{m_\pi^2 - m_\mu^2}\left(1 - \frac{m_\pi - m_\mu}{m_\pi + m_\mu}\right) \\
&= -2m_\mu\sqrt{m_\pi^2 - m_\mu^2}
\end{aligned}$$

Therefore

$$\mathfrak{M} = f_\pi k_\mu \cdot \bar{u}\gamma^\mu\frac{1-\gamma^5}{2}v = -f_\pi m_\mu\sqrt{m_\pi^2 - m_\mu^2}$$

Now suppose $\vec{\beta}$ to be z-axis, and the decay in z-x plane.

θ: the angle between $\vec{\alpha}$ and $\vec{\beta}$.

$$\vec{\beta} = \hat{z}$$

$$\vec{\alpha} = \alpha\cos\theta\hat{z} + \alpha\sin\theta\hat{x}\ ,$$

where $\alpha = \frac{p_\mu}{E_\mu + m_\mu} = \frac{\gamma\beta}{\gamma+1}$, i.e. $\alpha^2 = \frac{\gamma^2\beta^2}{(\gamma+1)^2} = \frac{\gamma-1}{\gamma+1}$

$$\begin{aligned}
\mathfrak{M} &= -\frac{f_\pi m_\mu}{2}\sqrt{E_\nu}\sqrt{E_\mu + m_\mu}\,\phi^\dagger(1 - \vec{\sigma}\cdot\vec{\beta})(1 + \vec{\sigma}\cdot\vec{\alpha})\chi \\
&= -\frac{f_\pi m_\mu}{2}\sqrt{E_\nu}\sqrt{E_\mu + m_\mu}\,\phi^\dagger(1-\sigma_3)(1 + \alpha\cos\theta\sigma_3 + \alpha\sin\theta\sigma_1)\chi \\
&= -f_\pi m_\mu\sqrt{E_\nu}\sqrt{E_\mu + m_\mu}\,\phi^\dagger\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+\alpha\cos\theta & \alpha\sin\theta \\ \alpha\sin\theta & 1-\alpha\cos\theta \end{pmatrix}\chi
\end{aligned}$$

Possible 2-spinor for ϕ is only $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, that is corresponding to the neutrino only having minus helicity. Then

$$\begin{aligned}
\mathfrak{M} &= -f_\pi m_\mu\sqrt{E_\nu}\sqrt{E_\mu + m_\mu}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1+\alpha\cos\theta & \alpha\sin\theta \\ \alpha\sin\theta & 1-\alpha\cos\theta \end{pmatrix}\chi \\
&= -f_\pi m_\mu\sqrt{E_\nu}\sqrt{E_\mu + m_\mu}\begin{pmatrix} \alpha\sin\theta & 1-\alpha\cos\theta \end{pmatrix}\chi
\end{aligned}$$

(To check the calculation)

In the above matrix element, we found

$$\chi = \frac{1}{\sqrt{\alpha^2\sin^2\theta + (1-\alpha\cos\theta)^2}}\begin{pmatrix} \alpha\sin\theta \\ 1-\alpha\cos\theta \end{pmatrix} = \frac{1}{\sqrt{1+\alpha^2 - 2\alpha\cos\theta}}\begin{pmatrix} \alpha\sin\theta \\ 1-\alpha\cos\theta \end{pmatrix}$$

i.e.

$$\mathfrak{M} = -f_\pi m_\mu\sqrt{E_\nu}\sqrt{E_\mu + m_\mu}\sqrt{1+\alpha^2 - 2\alpha\cos\theta}$$

Remind

$$m_\pi^2 = (p_\mu + p_\nu)^2 = m_\mu^2 + 2(E_\mu E_\nu - p_\mu p_\nu \cos\theta) = m_\mu^2 + 2E_\nu\{E_\mu - \alpha(E_\mu + m_\mu)\cos\theta\}$$

$$m_\pi^2 - m_\mu^2 = 2E_\nu\{E_\mu - (E_\mu + m_\mu)\alpha\cos\theta\} = E_\nu(E_\mu + m_\mu)\left(\frac{2E_\mu}{E_\mu + m_\mu} - 2\alpha\cos\theta\right)$$

Here use $1 + \alpha^2 = 1 + \frac{\gamma - 1}{\gamma + 1} = \frac{2\gamma}{\gamma + 1} = \frac{2E_\mu}{E_\mu + m_\mu}$

$$m_\pi^2 - m_\mu^2 = E_\nu(E_\mu + m_\mu)(1 + \alpha^2 - 2\alpha\cos\theta)$$

Therefore

$$\mathfrak{M} = -f_\pi m_\mu \sqrt{E_\nu}\sqrt{E_\mu + m_\mu}\sqrt{\frac{m_\pi^2 - m_\mu^2}{E_\nu(E_\mu + m_\mu)}} = -f_\pi m_\mu \sqrt{m_\pi^2 - m_\mu^2}$$

This is consistent with the calculation in the pion rest frame above.

For μ^+(anti-particle), since μ^+ is moving to the direction of θ w.r.t. z-axis(ν flight direction), two helicity eigen states are described as

$$\chi_\uparrow(\theta) = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}, \ \chi_\downarrow(\theta) = \begin{pmatrix} -\cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \end{pmatrix}$$

For each state, the amplitude is given by

$$\begin{aligned}\mathfrak{M}_\uparrow &\propto \begin{pmatrix} \alpha\sin\theta & 1 - \alpha\cos\theta \end{pmatrix}\begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} = -\alpha\sin\theta\sin\frac{\theta}{2} + (1 - \alpha\cos\theta)\cos\frac{\theta}{2} \\ &= -\alpha\sin\theta\sin\frac{\theta}{2} + \cos\frac{\theta}{2} - \alpha\cos\theta\cos\frac{\theta}{2} = -\alpha\cos\frac{\theta}{2} + \cos\frac{\theta}{2} \\ &= (1 - \alpha)\cos\frac{\theta}{2}\end{aligned}$$

$$\begin{aligned}\mathfrak{M}_\downarrow &\propto \begin{pmatrix} \alpha\sin\theta & 1 - \alpha\cos\theta \end{pmatrix}\begin{pmatrix} -\cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \end{pmatrix} = -\alpha\sin\theta\cos\frac{\theta}{2} - (1 - \alpha\cos\theta)\sin\frac{\theta}{2} \\ &= -\alpha\sin\theta\cos\frac{\theta}{2} - \sin\frac{\theta}{2} + \alpha\cos\theta\sin\frac{\theta}{2} = -\alpha\sin\frac{\theta}{2} - \sin\frac{\theta}{2} \\ &= -(1 + \alpha)\sin\frac{\theta}{2}\end{aligned}$$

The probability for each helicity state is

$$|\mathfrak{M}_\uparrow|^2 \propto \frac{2\gamma}{\gamma + 1}(1 - \beta)\cos^2\frac{\theta}{2}$$

$$|\mathfrak{M}_\downarrow|^2 \propto \frac{2\gamma}{\gamma + 1}(1 + \beta)\sin^2\frac{\theta}{2},$$

where β is the velocity of muon, since note that

$$(1 - \alpha)^2 = 1 - 2\alpha + \alpha^2 = \frac{\gamma + 1 - 2\gamma\beta + \gamma - 1}{\gamma + 1} = \frac{2\gamma}{\gamma + 1}(1 - \beta)$$

$$(1 + \alpha)^2 = 1 + 2\alpha + \alpha^2 = \frac{\gamma + 1 + 2\gamma\beta + \gamma - 1}{\gamma + 1} = \frac{2\gamma}{\gamma + 1}(1 + \beta)$$

(To confirm the calculation)

$$|\mathfrak{M}_\uparrow|^2 + |\mathfrak{M}_\downarrow|^2 \propto \frac{2\gamma}{\gamma+1}\left\{(1-\beta)\cos^2\frac{\theta}{2} + (1+\beta)\sin^2\frac{\theta}{2}\right\} = \frac{2\gamma}{\gamma+1}(1-\beta\cos\theta)$$

$$\begin{aligned}\sum |\mathfrak{M}_i|^2 &= f_\pi^2 m_\mu^2 E_\nu (E_\mu + m_\mu)\frac{2\gamma}{\gamma+1}(1-\beta\cos\theta) \\ &= f_\pi^2 m_\mu^2 E_\nu (E_\mu + m_\mu)\left(1+\alpha^2 - \frac{2\beta\gamma}{\gamma+1}\cos\theta\right) \\ &= f_\pi^2 m_\mu^2 E_\nu (E_\mu + m_\mu)\left(1+\alpha^2 - 2\alpha\cos\theta\right) \\ &= f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2)\end{aligned}$$

Define asymmetry of muon helicity

$$\mathcal{A} = \frac{|\mathfrak{M}_\uparrow|^2 - |\mathfrak{M}_\downarrow|^2}{|\mathfrak{M}_\uparrow|^2 + |\mathfrak{M}_\downarrow|^2} = \frac{\cos\theta - \beta}{1-\beta\cos\theta}$$

β: μ^+ velocity in Lab frame.

θ: opening angle between μ^+ and ν_μ in Lab frame.

Pion Decay in Lab. Frame

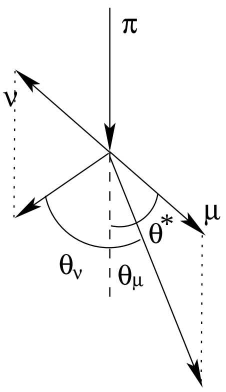
First, we consider $\pi^+ \to \mu^+\nu$ in π rest frame

$$p^* = \frac{\lambda^{1/2}(m_\pi^2, m_\mu^2, 0)}{2m_\pi} = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}$$

$$E_\mu^* = \frac{m_\pi^2 + m_\mu^2}{2m_\pi},\ E_\nu^* = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}$$

$p^* = 29.87\text{MeV}$, $E_\mu^* = 110.1\text{MeV}$, $1/\beta_\mu^* = 3.69$ for $m_\pi = 140\text{MeV}$, $m_\mu = 106\text{MeV}$

Then, boost the system along vertically downward.



Define θ_μ, θ_ν, and θ^* as above, then

$$\theta = \theta_\mu + \theta_\nu$$

Boosted by β_π, γ_π of the pion in Lab frame.

$$\begin{pmatrix} E \\ P_\parallel \end{pmatrix} = \begin{pmatrix} \gamma_\pi & \gamma_\pi\beta_\pi \\ \gamma_\pi\beta_\pi & \gamma_\pi \end{pmatrix}\begin{pmatrix} E^* \\ P_\parallel^* \end{pmatrix}$$

$$p_{\mu\parallel} = \gamma_\pi\beta_\pi E_\mu^* + \gamma_\pi p^*\cos\theta^* \qquad E_\mu = \gamma_\pi E_\mu^* + \gamma_\pi\beta_\pi p^*\cos\theta^*$$

$$p_{\nu\parallel} = \gamma_\pi\beta_\pi E_\nu^* - \gamma_\pi p^*\cos\theta^* = \gamma_\pi p^*(\beta_\pi - \cos\theta^*)$$

$$E_\nu = \gamma_\pi E^*_\nu - \gamma_\pi \beta_\pi p^* \cos\theta^* = \gamma_\pi p^* (1 - \beta_\pi \cos\theta^*)$$

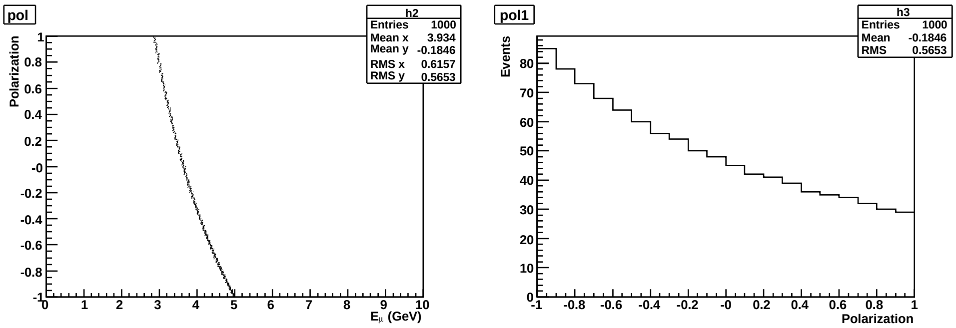
γ factor of muon, $\tan\theta_\mu$, and $\tan\theta_\nu$ in Lab frame can be described as

$$\gamma = \frac{\gamma_\pi E^*_\mu + \gamma_\pi \beta_\pi p^* \cos\theta^*}{m_\mu} = \frac{\gamma_\pi p^*}{m_\mu}(E^*_\mu / p^* + \beta_\pi \cos\theta^*) = \frac{\gamma_\pi p^*}{m_\mu}(1/\beta^*_\mu + \beta_\pi \cos\theta^*)$$

$$\tan\theta_\mu = \frac{p^* \sin\theta^*}{\gamma_\pi \beta_\pi E^*_\mu + \gamma_\pi p^* \cos\theta^*} = \frac{\sin\theta^*}{\gamma_\pi (\beta_\pi E^*_\mu / p^* + \cos\theta^*)} = \frac{\sin\theta^*}{\gamma_\pi (\beta_\pi / \beta^*_\mu + \cos\theta^*)}$$

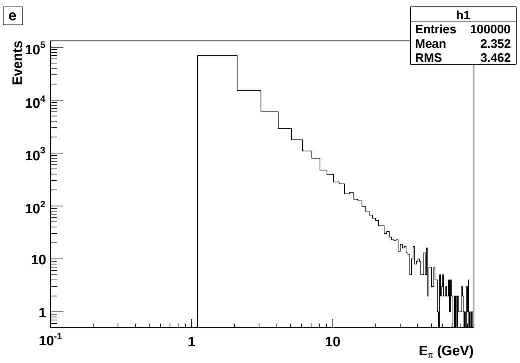
$$\tan\theta_\nu = \frac{p^* \sin\theta^*}{\gamma_\pi p^* (\beta_\pi - \cos\theta^*)} = \frac{\sin\theta^*}{\gamma_\pi (\beta_\pi - \cos\theta^*)}$$

Simulated results of $\mathcal{A}$ (Polarization) for $E_\pi = 5$ GeV and uniform distribution of $\cos\theta^*$ are shown in the following plots. Polarization of μ^+ as a function of E_μ in Lab frame (Left). The distribution of μ^+ polarization in Lab frame (Right).

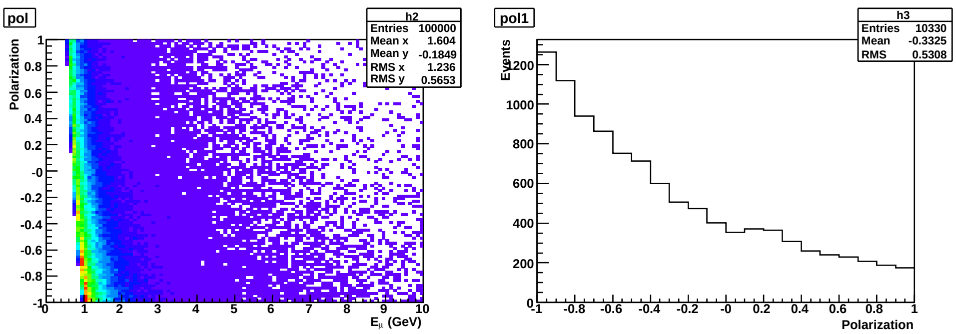


Assume E_π is distributed by

$$\frac{dN}{dE_\pi} \propto E_\pi^{-2.7} \text{ (for } E_\pi \text{ GeV)}$$



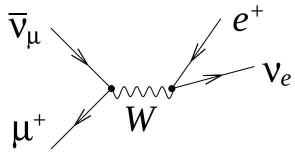
And if we select only the muons whose energies are within 2 GeV to 3 GeV, the distribution of μ^+ polarization in Lab frame is given by the following plot (right).



Muon is tend to found polarized to helicity minus. (P_+:P_-=0.33:0.67)

Muon Decay

4 Fermi V-A interaction



Matrix element of muon decay $\mu^+ \to e^+ \bar{\nu}_\mu \nu_e$ is proportional to

$$\mathfrak{M} \propto \{\bar{v}_{\mu^+}\gamma_\mu(1-\gamma^5)v_{\bar{\nu}_\mu}\}\{\bar{u}_{\nu_e}\gamma^\mu(1-\gamma^5)v_{e^+}\}.$$

In general

$$\begin{aligned}\bar{\psi}\gamma^\mu(1-\gamma^5)\phi &= \{\bar{\psi}\gamma^\mu(1-\gamma^5)\phi\}^T \\ &= -\phi^T\{\gamma^\mu(1-\gamma^5)\}^T\bar{\psi}^T \\ &= \bar{\phi}^C C\{\gamma^\mu(1-\gamma^5)\}^T C^{-1}\psi^C = -\bar{\phi}^C(1-\gamma^5)\gamma^\mu\psi^C \\ &= -\bar{\phi}^C\gamma^\mu(1+\gamma^5)\psi^C,\end{aligned}$$

where $\psi^C = C\bar{\psi}^T = i\gamma^2\psi^*$ and $C = i\gamma^2\gamma^0 = \begin{bmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{bmatrix}$

Using Fierz transformation

$$\begin{aligned}&\{\bar{\psi}_1\gamma_\mu(1-\gamma^5)\psi_2\}\{\bar{\psi}_3\gamma^\mu(1+\gamma^5)\psi_4\} \\ &\quad = -\{\bar{\psi}_1(1+\gamma^5)\psi_4\}\{\bar{\psi}_3(1-\gamma^5)\psi_2\} \\ &\quad + \frac{1}{2}\{\bar{\psi}_1\gamma_\mu(1+\gamma^5)\psi_4\}\{\bar{\psi}_3\gamma^\mu(1-\gamma^5)\psi_2\} \\ &\quad + \frac{1}{2}\{\bar{\psi}_1\gamma_\mu\gamma^5(1+\gamma^5)\psi_4\}\{\bar{\psi}_3\gamma^\mu\gamma^5(1-\gamma^5)\psi_2\} \\ &\quad + \{\bar{\psi}_1\gamma^5(1+\gamma^5)\psi_4\}\{\bar{\psi}_3\gamma^5(1-\gamma^5)\psi_2\} \\ &\quad = -2\{\bar{\psi}_1(1+\gamma^5)\psi_4\}\{\bar{\psi}_3(1-\gamma^5)\psi_2\}\end{aligned}$$

Therefore

$$\begin{aligned}\mathfrak{M} &\propto \{\bar{v}_{\mu^+}\gamma_\mu(1-\gamma^5)v_{\bar{\nu}_\mu}\}\{\bar{u}_{\nu_e}\gamma^\mu(1-\gamma^5)v_{e^+}\} \\ &= -\{\bar{v}_{\mu^+}\gamma_\mu(1-\gamma^5)v_{\bar{\nu}_\mu}\}\{\bar{v}^C_{e^+}\gamma^\mu(1+\gamma^5)u^C_{\nu_e}\} \\ &= 2\{\bar{v}_{\mu^+}(1+\gamma^5)u^C_{\nu_e}\}\{\bar{v}^C_{e^+}(1-\gamma^5)v_{\bar{\nu}_\mu}\}\end{aligned}$$

We don't observe the spins of dauter particles, so we take spin sum of them.

$$\begin{aligned}|\mathfrak{M}|^2 &\propto \sum_{s_{\nu_e}}\sum_{s_e, s_{\nu_\mu}} |\mathfrak{M}_{\mu\nu_e}|^2 |\bar{v}^C_{e^+}(1-\gamma^5)v_{\bar{\nu}_\mu}|^2 \\ &= \sum_{s_{\nu_e}} |\mathfrak{M}_{\mu\nu_e}|^2 \sum_{s_e, s_{\nu_\mu}} \mathrm{Tr}\{(1-\gamma^5)v_{\bar{\nu}_\mu}\bar{v}_{\bar{\nu}_\mu}(1+\gamma^5)v^C_{e^+}\bar{v}^C_{e^+}\} \\ &= \sum_{s_{\nu_e}} |\mathfrak{M}_{\mu\nu_e}|^2 \mathrm{Tr}\Big\{(1-\gamma^5)\not{p}_{\bar{\nu}_\mu}(1+\gamma^5)\Big(-\gamma^2\not{p}^*_{e^+}\gamma^2\Big)\Big\} \\ &= \sum_{s_{\nu_e}} |\mathfrak{M}_{\mu\nu_e}|^2 \mathrm{Tr}\Big\{(1-\gamma^5)\not{p}_{\bar{\nu}_\mu}(1+\gamma^5)\not{p}_{e^+}\Big\} \\ &= \sum_{s_{\nu_e}} |\mathfrak{M}_{\mu\nu_e}|^2 \mathrm{Tr}\Big\{2\not{p}_{\bar{\nu}_\mu}\not{p}_{e^+} - 2\gamma^5\not{p}_{\bar{\nu}_\mu}\not{p}_{e^+}\Big\} \\ &= 8\sum_{s_{\nu_e}} |\mathfrak{M}_{\mu\nu_e}|^2 (p_{\nu_\mu}\cdot p_{e^+})\end{aligned}$$

In the muon rest frame, define spinors as

$$v_{\mu^+} = \sqrt{2m_\mu}\begin{pmatrix} 0 \\ \chi_{\mu^+} \end{pmatrix} \qquad u_{\nu_e} = \sqrt{E_{\nu_e}}\begin{pmatrix} \phi_{\nu_e} \\ \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\nu}}_{\boldsymbol{e}}\phi_{\nu_e} \end{pmatrix}$$

$$u_{\nu_e}^C = i\gamma^2 u_{\nu_e}^* \propto \begin{bmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{bmatrix}\begin{pmatrix} \phi_{\nu_e}^* \\ \boldsymbol{\sigma}^*\cdot\hat{\boldsymbol{\nu}}_{\boldsymbol{e}}\phi_{\nu_e}^* \end{pmatrix} = \begin{pmatrix} i\sigma_2\boldsymbol{\sigma}^*\cdot\hat{\boldsymbol{\nu}}_{\boldsymbol{e}}\phi_{\nu_e}^* \\ -i\sigma_2\phi_{\nu_e}^* \end{pmatrix} = -\begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\nu}}_{\boldsymbol{e}} i\sigma_2\phi_{\nu_e}^* \\ i\sigma_2\phi_{\nu_e}^* \end{pmatrix}$$

$$\begin{aligned}
\mathfrak{M}_{\mu\nu_e} &= \sqrt{2m_\mu E_{\nu_e}}\{\bar{v}_{\mu^+}(1+\gamma^5)u_{\nu_e}^C\} \\
&= -\sqrt{2m_\mu E_{\nu_e}}\begin{pmatrix} 0 & -\chi_{\mu^+}^\dagger \end{pmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\nu}}_{\boldsymbol{e}} i\sigma_2\phi_{\nu_e}^* \\ i\sigma_2\phi_{\nu_e}^* \end{pmatrix} \\
&= \sqrt{2m_\mu E_{\nu_e}}\chi_{\mu^+}^\dagger(1+\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\nu}}_{\boldsymbol{e}})i\sigma_2\phi_{\nu_e}^*
\end{aligned}$$

Assume the decay to be in z-x plane, and now μ^+ is polarized to z-axis. Also define the flight angles of decay products as θ_{ν_e}, θ_e, and θ_{ν_μ}, respectively

$$\chi_{\mu^+} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned}
1+\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\nu}}_{\boldsymbol{e}} &= 1+\sigma_3\cos\theta_{\nu_e}+\sigma_1\sin\theta_{\nu_e} = \begin{bmatrix} 1+\cos\theta_{\nu_e} & \sin\theta_{\nu_e} \\ \sin\theta_{\nu_e} & 1-\cos\theta_{\nu_e} \end{bmatrix} \\
&= 2\begin{bmatrix} \cos^2\frac{\theta_{\nu_e}}{2} & \sin\frac{\theta_{\nu_e}}{2}\cos\frac{\theta_{\nu_e}}{2} \\ \sin\frac{\theta_{\nu_e}}{2}\cos\frac{\theta_{\nu_e}}{2} & \sin^2\frac{\theta_{\nu_e}}{2} \end{bmatrix}
\end{aligned}$$

This means $\mathfrak{M}_{\mu\nu_e} = 0$ for

$$i\sigma_2\phi_{\nu_e}^* = \begin{pmatrix} -\sin\frac{\theta_{\nu_e}}{2} \\ \cos\frac{\theta_{\nu_e}}{2} \end{pmatrix}$$

Hence only allows

$$i\sigma_2\phi_{\nu_e}^* = \begin{pmatrix} \cos\frac{\theta_{\nu_e}}{2} \\ \sin\frac{\theta_{\nu_e}}{2} \end{pmatrix}$$

i.e.

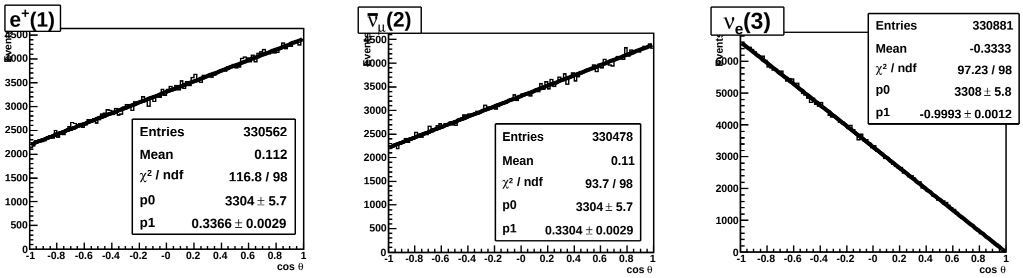
$$\phi_{\nu_e} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1}\begin{pmatrix} \cos\frac{\theta_{\nu_e}}{2} \\ \sin\frac{\theta_{\nu_e}}{2} \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} \cos\frac{\theta_{\nu_e}}{2} \\ \sin\frac{\theta_{\nu_e}}{2} \end{pmatrix} = \begin{pmatrix} -\sin\frac{\theta_{\nu_e}}{2} \\ \cos\frac{\theta_{\nu_e}}{2} \end{pmatrix}$$

which means ν_e has helicity minus.

$$\begin{aligned}
\mathfrak{M}_{\mu\nu_e} &= \sqrt{2m_\mu E_{\nu_e}}\chi_{\mu^+}^\dagger(1+\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\nu}}_{\boldsymbol{e}})i\sigma_2\phi_{\nu_e}^* \\
&= 2\sqrt{2m_\mu E_{\nu_e}}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{bmatrix} \cos^2\frac{\theta_{\nu_e}}{2} & \sin\frac{\theta_{\nu_e}}{2}\cos\frac{\theta_{\nu_e}}{2} \\ \sin\frac{\theta_{\nu_e}}{2}\cos\frac{\theta_{\nu_e}}{2} & \sin^2\frac{\theta_{\nu_e}}{2} \end{bmatrix}\begin{pmatrix} \cos\frac{\theta_{\nu_e}}{2} \\ \sin\frac{\theta_{\nu_e}}{2} \end{pmatrix} \\
&= 2\sqrt{2m_\mu E_{\nu_e}}\begin{pmatrix} \sin\frac{\theta_{\nu_e}}{2}\cos\frac{\theta_{\nu_e}}{2} & \sin^2\frac{\theta_{\nu_e}}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta_{\nu_e}}{2} \\ \sin\frac{\theta_{\nu_e}}{2} \end{pmatrix} \\
&= 2\sqrt{2m_\mu E_{\nu_e}}\sin\frac{\theta_{\nu_e}}{2} = 2\sqrt{m_\mu E_{\nu_e}(1-\cos\theta_{\nu_e})}
\end{aligned}$$

$$|\mathfrak{M}|^2 \propto 8\sum_{s_{\nu_e}}|\mathfrak{M}_{\mu\nu_e}|^2(p_{\nu_\mu}\cdot p_{e^+}) = 32m_\mu E_{\nu_e}(1-\cos\theta_{\nu_e})(p_{\nu_\mu}\cdot p_{e^+})$$

For $\mu^+ \to e^+\bar{\nu}_\mu\nu_e$ decay, performed a simulation of 3-body decays with the matrix element above, then obtained the following differential decay rates:



$$\frac{d\Gamma}{\mathrm{dcos}\theta_e} = \frac{1+\alpha_e\cos\theta_e}{2}\ \text{, where } \alpha_e \sim 0.33$$

A positron from a μ^+ decay tends to be emitted to the direction of the muon polarization.

We define the probability that positron from μ^+ is emitted to the same (opposite) semisphere as the muon polarization as $p_+(p_-)$.

$$p_+ = \int_0^1 \frac{1+\alpha_e x}{2}dx = \frac{1}{2}\Big[x + \frac{\alpha_e}{2}x^2\Big]_0^1 = \frac{1}{2}\Big(1+\frac{\alpha_e}{2}\Big) = 0.58$$

$$p_- = \frac{1}{2}\Big(1-\frac{\alpha_e}{2}\Big) = 0.42$$

In case of a muon falling vertically and stopped in a material with conserving its spin, the probability that the positron is emitted upward(downward) semisphere is

$$P_{\mathrm{down}} = P_+ \times p_+ + P_- \times p_- = .4728$$

$$P_{\mathrm{up}} = P_+ \times p_- + P_- \times p_+ = .5278$$

About 5% asymmetry will be expected.

[EOF]