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# Muon Precession

## Magnetic Moment of Dirac Particle

Direc eq. in electro-magnetic field $A^\mu = (\phi, A_i)$

$$[\gamma^\mu(i\partial_\mu - qA_\mu) - m]\psi = 0 \text{ , where } \partial_\mu = (\partial_t, \partial_i)$$

$$\left(\gamma^0 i\partial_t + \gamma^i i\partial_i - q\gamma^0\phi + q\gamma^i A_i - m\right)\psi = 0$$

$$(i\partial_t - q\phi)\psi = \left\{\gamma^0\gamma^i(-i\partial_i - qA_i) + \gamma^0 m\right\}\psi$$

Note $\gamma^0\gamma^i = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} = \begin{pmatrix} & \sigma_i \\ \sigma_i & \end{pmatrix}$. Then we obtain

$$(i\partial_t - q\phi)\psi = \begin{pmatrix} m & \sigma_i(-i\partial_i - qA_i) \\ \sigma_i(-i\partial_i - qA_i) & -m \end{pmatrix}\psi$$

$$(i\partial_t - q\phi - m)u = \sigma_i(-i\partial_i - qA_i)v$$

$$(i\partial_t - q\phi + m)v = \sigma_i(-i\partial_i - qA_i)u$$

Therefore

$$\begin{aligned}(i\partial_t - q\phi + m)(i\partial_t - q\phi - m)u &= (i\partial_t - q\phi + m)\sigma_i(-i\partial_i - qA_i)v \\ &= \sigma_i\{(-i\partial_i - qA_i)(i\partial_t - q\phi + m) - iq[\![\partial_t A_i]\!] - iq[\![\partial_i\phi]\!]\}v \\ &= \sigma_i\{(-i\partial_i - qA_i)(i\partial_t - q\phi + m) + iqE_i\}v \\ &= \sigma_i(-i\partial_i - qA_i)\sigma_j(-i\partial_j - qA_j)u + iq\sigma_i E_i v\end{aligned}$$

Remind $\partial f = [\![\partial f]\!] + f\partial$, $f\partial = \partial f - [\![\partial f]\!]$, and $E_i = -\partial_i\phi - \partial_t A_i$

Suppose $E_i = 0$

$$\begin{aligned}(i\partial_t - q\phi + m)(i\partial_t - q\phi - m)u &= \sigma_i\sigma_j(-i\partial_i - qA_i)(-i\partial_j - qA_j)u \\ &= (\delta_{ij} + i\sigma_k\epsilon_{ijk})(-i\partial_i - qA_i)(-i\partial_j - qA_j)u \\ &= \left(-i\vec{\nabla} - q\vec{A}\right)^2 u - q\sigma_k\epsilon_{ijk}\{\partial_i(A_j u) + A_i\partial_j u\} \\ &= \left(-i\vec{\nabla} - q\vec{A}\right)^2 u - q\sigma_k\epsilon_{ijk}[\![\partial_i A_j]\!]u \\ &= \left(-i\vec{\nabla} - q\vec{A}\right)^2 u - q\vec{\sigma}\cdot\vec{B}u\end{aligned}$$

For non-relativistic particle, suppose $(i\partial_t - q\phi + m)u \sim 2mu$

$$2m(i\partial_t - q\phi - m)u = \left(-i\vec{\nabla} - q\vec{A}\right)^2 u - q\vec{\sigma}\cdot\vec{B}u$$

$$(i\partial_t - q\phi - m)u = \frac{1}{2m}\left(-i\vec{\nabla} - q\vec{A}\right)^2 u - \frac{q}{2m}\vec{\sigma}\cdot\vec{B}u$$

$$i\partial_t u = \left\{\frac{1}{2m}\left(-i\vec{\nabla} - q\vec{A}\right)^2 - \frac{q}{2m}\vec{\sigma}\cdot\vec{B} + q\phi + m\right\}u$$

Found the potential term by $\vec{B}$ to be $U = -\vec{\mu}\cdot\vec{B} = -\frac{q}{2m}\vec{\sigma}\cdot\vec{B}$, i.e.

$$\vec{\mu} = \frac{q}{2m}\vec{\sigma} = g\frac{q}{2m}\vec{s} \text{ ,}$$

where $\vec{s} = \vec{\sigma}/2$, and $g = 2$

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If the magnetic moment and the spin originate from a rotational movement of charge $q$ and mass $m$,

$$\vec{\mu} = \frac{1}{2}\int_C \vec{x} \times \vec{J}(\vec{x})dl,$$

where $C$ is a circle with radius $r$, and $\vec{J} = \rho\vec{v} = \frac{q}{2\pi r}\vec{v}$

$$\begin{aligned}\vec{\mu} &= \frac{1}{2}\int_C \vec{x} \times \vec{J}(\vec{x})dl = \frac{1}{2}\int_C \vec{x} \times \frac{q}{2\pi r}\vec{v}dl \\ &= \frac{q}{2m}\int_C \frac{m}{2\pi r}\vec{x} \times \vec{v}dl \\ &= \frac{q}{2m}\vec{L}\end{aligned}$$

Hence, found

$$g = 1$$

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## Precession

Now we take $q = -e$

$$i\frac{\partial u}{\partial t} = Hu = -\vec{\mu} \cdot \vec{B}u = g\frac{e}{2m}\frac{\vec{\sigma}}{2} \cdot \vec{B}u$$

Suppose $\vec{B} = (B, 0, 0)$, $u = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = -\begin{pmatrix} 0 & i\omega/2 \\ i\omega/2 & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} \qquad \omega = \frac{geB}{2m}$$

$$\begin{pmatrix} \ddot{\xi} \\ \ddot{\eta} \end{pmatrix} = \begin{pmatrix} 0 & i\omega/2 \\ i\omega/2 & 0 \end{pmatrix}^2\begin{pmatrix} \xi \\ \eta \end{pmatrix} = -\left(\frac{\omega}{2}\right)^2\begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \mathrm{A}\cos\{\frac{\omega}{2}(t - t_0)\} \\ -i\,\mathrm{A}\sin\{\frac{\omega}{2}(t - t_0)\} \end{pmatrix}$$

Requiring $u^\dagger u = 1$ and $u(t = 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos(\frac{\omega}{2}t) \\ -i\sin(\frac{\omega}{2}t) \end{pmatrix}$$

## For muon case

Use $m = 106\,\mathrm{MeV}/c^2$,

$$\omega/B = \frac{e}{m} = \frac{(3 \times 10^8)^2}{106 \times 10^6} = 8.5 \times 10^8\,\mathrm{rad}/(s \cdot T) = 13.5\mathrm{kHz/gauss}\ (1T = 10^4\mathrm{gauss})$$

The period $T$

$$T \cdot B = 74\mu s \cdot \mathrm{gauss}$$

$$T = 1\mu s \text{ for } \sim 75\mathrm{gauss}$$

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