Fierz transformations

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Considering the following 4-fermi interaction

$$M_i = (\bar{\psi}_1 \Gamma_i \psi_2)(\bar{\psi}_3 \Gamma_i \psi_4),$$

where $\Gamma_i = S(1),\ V^\mu(\gamma^\mu),\ T^{\rho\sigma}(\sigma^{\rho\sigma} = i/2(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho) = i\gamma^\rho\gamma^\sigma,\ \rho < \sigma),\ A^\nu(\gamma^\nu\gamma^5),\ P(\gamma^5)$

i.e. $\Gamma_i = 1, \gamma^0, \gamma^1, \gamma^2, \gamma^3, \sigma^{01}, \sigma^{02}, \sigma^{03}, \sigma^{12}, \sigma^{13}, \sigma^{23}, \gamma^0\gamma^5, \gamma^1\gamma^5, \gamma^2\gamma^5, \gamma^3\gamma^5, \gamma^5$

$$\begin{aligned} M_i &= (\bar{\psi}_1 \Gamma_i \psi_2)(\bar{\psi}_3 \Gamma_i \psi_4) = \sum_{\alpha,\beta} (\bar{\psi}_{1\alpha} \Gamma_{i\alpha\beta} \psi_{2\beta}) \sum_{\gamma,\delta} (\bar{\psi}_{3\gamma} \Gamma_{i\gamma\delta} \psi_{4\delta}) \\ &= \sum_{\alpha,\beta,\gamma,\delta} \Gamma_{i\alpha\beta} \Gamma_{i\gamma\delta} \bar{\psi}_{1\alpha} \psi_{2\beta} \bar{\psi}_{3\gamma} \psi_{4\delta} \\ &= -\sum_{\alpha,\beta,\gamma,\delta} \Gamma_{i\alpha\beta} \Gamma_{i\gamma\delta} \bar{\psi}_{1\alpha} \psi_{4\delta} \bar{\psi}_{3\gamma} \psi_{2\beta} \end{aligned}$$

Here suppose $\Gamma_{i\alpha\beta}\Gamma_{i\gamma\delta}$ as 4×4 matrix with index i and $\alpha\delta$, i.e. $\Theta_{i\alpha\delta} = (\Theta_{i\alpha\delta})_{\gamma\beta}$

Matrix Θ can be represent by 16 independent matrix Γ_j

$$\Gamma_{i\alpha\beta}\Gamma_{i\gamma\delta} = -\sum_j \Lambda^j_{i\alpha\delta} \Gamma_{j\gamma\beta}$$

Next suppose $\Lambda^j_{i\alpha\delta}$ as 4×4 matrix with index i and j, i.e. $\Lambda^j_i = (\Lambda^j_i)_{\alpha\delta}$

$$\Lambda^j_{i\alpha\delta} = \sum_k \lambda_{ijk} \Gamma_{k\alpha\delta}$$

Then

$$\Gamma_{i\alpha\beta}\Gamma_{i\gamma\delta} = -\sum_j \sum_k \lambda_{ijk} \Gamma_{k\alpha\delta} \Gamma_{j\gamma\beta}$$

Multiply $\Gamma_{l\beta\gamma}$ and take sum over β and γ

$$\Gamma_{i\alpha\beta}\Gamma_{l\beta\gamma}\Gamma_{i\gamma\delta} = -\sum_j \sum_k \lambda_{ijk} \Gamma_{k\alpha\delta} \Gamma_{j\gamma\beta} \Gamma_{l\beta\gamma}$$

$$\Gamma_i \Gamma_l \Gamma_i = -\sum_j \sum_k \lambda_{ijk} \Gamma_k \mathrm{Tr}(\Gamma_j \Gamma_l)$$

Note

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$$

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4[g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}]$$

$$\mathrm{Tr}[\gamma^1\gamma^2 \cdots \gamma^{2n+1}] = 0$$

$$\mathrm{Tr}[\gamma^5] = \mathrm{Tr}[\gamma^5\gamma^\mu] = \mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu] = \mathrm{Tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho] = 0$$

For $\mathrm{Tr}(\Gamma_i\Gamma_j) = \mathrm{Tr}(\Gamma_j\Gamma_i)$, obtain

$\mathrm{Tr}(\Gamma_i\Gamma_j)$	S	$V^{\mu'}$	$T^{\rho'\sigma'}$	$A^{\nu'}$	P
S	4	0	0	0	0
V^μ		$4g^{\mu\mu'}$	0	0	0
$T^{\rho\sigma}$			$4g^{\rho\rho'}g^{\sigma\sigma'}$	0	0
A^ν				$-4g^{\nu\nu'}$	0
P					4

$$\begin{aligned} \mathrm{Tr}(\sigma^{\rho\sigma}\sigma^{\rho'\sigma'}) &= -\mathrm{Tr}(\gamma^\rho\gamma^\sigma\gamma^{\rho'}\gamma^{\sigma'}) \qquad (\rho < \sigma,\ \rho' < \sigma') \\ &= -4(g^{\rho\sigma'}g^{\sigma\rho'} - g^{\rho\rho'}g^{\sigma\sigma'}) \\ &= 4g^{\rho\rho'}g^{\sigma\sigma'} \qquad (\text{if } \sigma = \rho',\ \rho < \sigma = \rho' < \sigma') \end{aligned}$$

Therefore $\mathrm{Tr}(\Gamma_i\Gamma_j)=4\epsilon_i\delta_{ij}$, where

$$\epsilon_i \equiv \begin{cases} 1 & (\Gamma_i = 1, \gamma^0, \sigma^{12}, \sigma^{13}, \sigma^{23}, \gamma^1\gamma^5, \gamma^2\gamma^5, \gamma^3\gamma^5, \gamma^5) \\ -1 & (\text{otherwise}) \end{cases}$$

Then

$$\Gamma_i\Gamma_l\Gamma_i = -4\sum_j\sum_k \lambda_{ijk}\Gamma_k\epsilon_j\delta_{jl} = -4\sum_k \lambda_{ilk}\epsilon_l\Gamma_k$$

Multiply Γ_j from right and take the trace

$$\mathrm{Tr}(\Gamma_i\Gamma_l\Gamma_i\Gamma_j) = -4\sum_k \lambda_{ilk}\epsilon_l \mathrm{Tr}(\Gamma_k\Gamma_j) = -16\sum_k \lambda_{ilk}\epsilon_l\epsilon_k\delta_{kj} = -16\lambda_{ilj}\epsilon_l\epsilon_j$$

Therefore we obtain

$$\lambda_{ijk} = -\frac{1}{16}\epsilon_j\epsilon_k \mathrm{Tr}(\Gamma_i\Gamma_j\Gamma_i\Gamma_k)$$

Since $\gamma^\mu\gamma^\nu = \begin{cases} \gamma^\nu\gamma^\mu & (\mu=\nu) \\ -\gamma^\nu\gamma^\mu & (\mu\neq\nu) \end{cases}$ and $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$,

$$\Gamma_i\Gamma_j = \eta_{ij}\Gamma_j\Gamma_i \text{ , where } \eta_{ij} = \pm 1$$

Define $n(\mu,\nu;\rho,\sigma)$ as a number of pairs between μ,ν and ρ, σ

η_{ij}	S	$V^{\mu'}$	$T^{\rho'\sigma'}$	$A^{\nu'}$	P
S	$+$	$+$	$+$	$+$	$+$
V^μ		$-(-1)^{n(\mu;\mu')}$	$(-1)^{n(\mu;\rho',\sigma')}$	$(-1)^{n(\mu;\nu')}$	$-$
$T^{\rho\sigma}$			$(-1)^{n(\rho,\sigma;\rho',\sigma')}$	$(-1)^{n(\rho,\sigma;\nu')}$	$+$
A^ν				$-(-1)^{n(\nu;\nu')}$	$-$
P					$+$

and

$$\Gamma_S\Gamma_S = 1$$

$$\Gamma_{V^\mu}\Gamma_{V^\mu} = \gamma^\mu\gamma^\mu = g^{\mu\mu}$$

$$\Gamma_{T^{\rho\sigma}}\Gamma_{T^{\rho\sigma}} = -\gamma^\rho\gamma^\sigma\gamma^\rho\gamma^\sigma = \gamma^\rho\gamma^\sigma\gamma^\sigma\gamma^\rho = g^{\rho\rho}g^{\sigma\sigma} \ (\rho<\sigma)$$

$$\Gamma_{A^\nu}\Gamma_{A^\nu} = \gamma^\nu\gamma^5\gamma^\nu\gamma^5 = -\gamma^\nu\gamma^5\gamma^5\gamma^\nu = -g^{\nu\nu}$$

$$\Gamma_P\Gamma_P = \gamma^5\gamma^5 = 1$$

i.e., $\Gamma_i\Gamma_i = \epsilon_i$, then

$$\begin{aligned}\lambda_{ijk} &= -\frac{1}{16}\epsilon_j\epsilon_k\mathrm{Tr}(\Gamma_i\Gamma_j\Gamma_i\Gamma_k) = -\frac{1}{16}\epsilon_j\epsilon_k\eta_{ij}\mathrm{Tr}(\Gamma_j\Gamma_i\Gamma_i\Gamma_k) = -\frac{1}{16}\epsilon_i\epsilon_j\epsilon_k\eta_{ij}\mathrm{Tr}(\Gamma_j\Gamma_k) \\ &= -\frac{1}{4}\epsilon_i\epsilon_k\eta_{ij}\delta_{jk}\end{aligned}$$

Therefore,

$$\Gamma_{i\alpha\beta}\Gamma_{i\gamma\delta} = -\sum_j\sum_k \lambda_{ijk}\Gamma_{k\alpha\delta}\Gamma_{j\gamma\beta} = -\sum_j \lambda_{ij}\Gamma_{j\alpha\delta}\Gamma_{j\gamma\beta} \text{ ,}$$

where $\lambda_{ij} \equiv -\frac{1}{4}\epsilon_i\epsilon_j\eta_{ij}$, and

$$\begin{aligned}(\bar\psi_1\Gamma_i\psi_2)(\bar\psi_3\Gamma_i\psi_4) &= -\sum_{\alpha,\beta,\gamma,\delta}\Gamma_{i\alpha\beta}\Gamma_{i\gamma\delta}\bar\psi_{1\alpha}\psi_{4\delta}\bar\psi_{3\gamma}\psi_{2\beta} \\ &= \sum_{\alpha,\beta,\gamma,\delta}\sum_j \lambda_{ij}\Gamma_{j\alpha\delta}\Gamma_{j\gamma\beta}\bar\psi_{1\alpha}\psi_{4\delta}\bar\psi_{3\gamma}\psi_{2\beta} \text{ ,} \\ &= \sum_j \lambda_{ij}(\bar\psi_1\Gamma_j\psi_4)(\bar\psi_3\Gamma_j\psi_2)\end{aligned}$$

Here $(\Gamma)(\Gamma)$ means $(\bar{\psi}_a\Gamma\psi_b)(\bar{\psi}_c\Gamma\psi_d)$

$$(\gamma^\mu)(\gamma_\mu) = (\gamma^0)(\gamma^0) - (\gamma^1)(\gamma^1) - (\gamma^2)(\gamma^2) - (\gamma^3)(\gamma^3) = \sum_{i=V^\mu} \epsilon_i(\Gamma_i)(\Gamma_i)$$

$$(\sigma^{\rho\sigma})(\sigma_{\rho\sigma}) = 2\sum_{i=T^{\rho\sigma}} \epsilon_i(\Gamma_i)(\Gamma_i)$$

$$(\gamma^\nu\gamma^5)(\gamma_\nu\gamma^5) = -\sum_{i=A^\nu} \epsilon_i(\Gamma_i)(\Gamma_i)$$

Considering above

$$\begin{aligned}(\bar{\psi}_1\psi_2)(\bar{\psi}_3\psi_4) &= \sum_j \lambda_{Sj}(\bar{\psi}_1\Gamma_j\psi_4)(\bar{\psi}_3\Gamma_j\psi_2) \\ &= -\frac{1}{4}\sum_j \epsilon_j(\bar{\psi}_1\Gamma_j\psi_4)(\bar{\psi}_3\Gamma_j\psi_2) \\ &= -\frac{1}{4}\left\{(1)(1) + (\gamma^\mu)(\gamma_\mu) + \frac{1}{2}(\sigma^{\rho\sigma})(\sigma_{\rho\sigma}) - (\gamma^\nu\gamma^5)(\gamma_\nu\gamma^5) + (\gamma^5)(\gamma^5)\right\}\end{aligned},$$

where $(\Gamma)(\Gamma)$ means $(\bar{\psi}_1\Gamma\psi_4)(\bar{\psi}_3\Gamma\psi_2)$

$$\begin{aligned}(\bar{\psi}_1\gamma^\mu\psi_2)(\bar{\psi}_3\gamma_\mu\psi_4) &= \sum_{i=V^\mu} \epsilon_i(\bar{\psi}_1\Gamma_i\psi_2)(\bar{\psi}_3\Gamma_i\psi_4) \\ &= \sum_{i=V^\mu} \epsilon_i \sum_j \lambda_{ij}\,(\bar{\psi}_1\Gamma_j\psi_4)(\bar{\psi}_3\Gamma_j\psi_2) \\ &= -\frac{1}{4}\sum_{i=V^\mu}\sum_j \epsilon_j\eta_{ij}(\bar{\psi}_1\Gamma_j\psi_4)(\bar{\psi}_3\Gamma_j\psi_2) \\ &= -\frac{1}{4}\sum_j \epsilon_j(\bar{\psi}_1\Gamma_j\psi_4)(\bar{\psi}_3\Gamma_j\psi_2)\sum_{i=V^\mu}\eta_{ij}\end{aligned}$$

$$\sum_{i=V^\mu}\eta_{ij} = \begin{cases} 4 & (j=S) \\ -2 & (j=V^{\mu'}) \\ 0 & (j=T^{\rho'\sigma'}) \\ 2 & (j=A^{\nu'}) \\ -4 & (j=P) \end{cases}$$

$$(\bar{\psi}_1\gamma^\mu\psi_2)(\bar{\psi}_3\gamma_\mu\psi_4) = -\frac{1}{4}\{4(1)(1) - 2(\gamma^\mu)(\gamma_\mu) - 2(\gamma^\nu\gamma^5)(\gamma_\nu\gamma^5) - 4(\gamma^5)(\gamma^5)\}$$

$$\begin{aligned}(\bar{\psi}_1\sigma^{\rho\sigma}\psi_2)(\bar{\psi}_3\sigma_{\rho\sigma}\psi_4) &= 2\sum_{i=T^{\rho\sigma}} \epsilon_i(\bar{\psi}_1\Gamma_i\psi_2)(\bar{\psi}_3\Gamma_i\psi_4) \\ &= -\frac{1}{2}\sum_j \epsilon_j(\bar{\psi}_1\Gamma_j\psi_4)(\bar{\psi}_3\Gamma_j\psi_2)\sum_{i=T^{\rho\sigma}}\eta_{ij}\end{aligned}$$

$$\sum_{i=T^{\rho\sigma}}\eta_{ij} = \begin{cases} 6 & (j=S) \\ 0 & (j=V^{\mu'}) \\ -2 & (j=T^{\rho'\sigma'}) \\ 0 & (j=A^{\nu'}) \\ 6 & (j=P) \end{cases}$$

$$(\bar{\psi}_1\sigma^{\rho\sigma}\psi_2)(\bar{\psi}_3\sigma_{\rho\sigma}\psi_4) = -\frac{1}{2}\{6(1)(1) - (\sigma^{\rho\sigma})(\sigma_{\rho\sigma}) + 6(\gamma^5)(\gamma^5)\}$$

$$\begin{aligned}(\bar{\psi}_1\gamma^\nu\gamma^5\psi_2)(\bar{\psi}_3\gamma_\nu\gamma^5\psi_4) &= -\sum_{i=A^\nu} \epsilon_i(\bar{\psi}_1\Gamma_i\psi_2)(\bar{\psi}_3\Gamma_i\psi_4) \\ &= \frac{1}{4}\sum_j \epsilon_j(\bar{\psi}_1\Gamma_j\psi_4)(\bar{\psi}_3\Gamma_j\psi_2)\sum_{i=A^\nu}\eta_{ij}\end{aligned}$$

$$\sum_{i=A^\nu} \eta_{ij} = \begin{cases} 4 & (j=S) \\ 2 & (j=V^{\mu'}) \\ 0 & (j=T^{\rho'\sigma'}) \\ -2 & (j=A^{\nu'}) \\ -4 & (j=P) \end{cases}$$

$$(\bar{\psi}_1\gamma^\nu\gamma^5\psi_2)(\bar{\psi}_3\gamma_\nu\gamma^5\psi_4) = \frac{1}{4}\{4(1)(1) + 2(\gamma^\mu)(\gamma_\mu) + 2(\gamma^\nu\gamma^5)(\gamma_\nu\gamma^5) - 4(\gamma^5)(\gamma^5)\}$$

$$\begin{aligned}(\bar{\psi}_1\gamma^5\psi_2)(\bar{\psi}_3\gamma^5\psi_4) &= \sum_j \lambda_{Pj}(\bar{\psi}_1\Gamma_j\psi_4)(\bar{\psi}_3\Gamma_j\psi_2) \\ &= -\frac{1}{4}\sum_j \epsilon_j\eta_{Pj}(\bar{\psi}_1\Gamma_j\psi_4)(\bar{\psi}_3\Gamma_j\psi_2) \\ &= -\frac{1}{4}\left\{(1)(1) - (\gamma^\mu)(\gamma_\mu) + \frac{1}{2}(\sigma^{\rho\sigma})(\sigma_{\rho\sigma}) + (\gamma^\nu\gamma^5)(\gamma_\nu\gamma^5) + (\gamma^5)(\gamma^5)\right\}\end{aligned}$$

Finally, if we define λ_{ij} by

$$(\bar{\psi}_1\Lambda_i\psi_2)(\bar{\psi}_3\Lambda_i\psi_4) = \sum_j \lambda_{ij}(\bar{\psi}_1\Lambda_j\psi_4)(\bar{\psi}_3\Lambda_j\psi_2),$$

where $\Lambda_i = S(1),\ V(\gamma^\mu),\ T(\sigma^{\rho\sigma}),\ A(\gamma^\nu\gamma^5),\ P(\gamma^5)$

λ_{ij}	S	V	T	A	P
S	$-1/4$	$-1/4$	$-1/8$	$1/4$	$-1/4$
V	-1	$1/2$	0	$1/2$	1
T	-3	0	$1/2$	0	-3
A	1	$1/2$	0	$1/2$	-1
P	$-1/4$	$1/4$	$-1/8$	$-1/4$	$-1/4$

Applications

$$\begin{aligned}\bar{\psi}\gamma^\mu(1-\gamma^5)\phi &= \{\bar{\psi}\gamma^\mu(1-\gamma^5)\phi\}^T \\ &= -\phi^T\{\gamma^\mu(1-\gamma^5)\}^T\bar{\psi}^T = \bar{\phi}^C C\{\gamma^\mu(1-\gamma^5)\}^T C^{-1}\psi^C \\ &= \bar{\phi}^C(1-\gamma^5)C\{\gamma^\mu\}^T C^{-1}\psi^C = -\bar{\phi}^C(1-\gamma^5)\gamma^\mu\psi^C = -\bar{\phi}^C\gamma^\mu(1+\gamma^5)\psi^C\end{aligned}$$

where $\psi^C = C\bar{\psi}^T = i\gamma^2\psi^*$ $\qquad C = i\gamma^2\gamma^0 = \begin{bmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{bmatrix}$

$$\begin{aligned} & \{\bar{\psi}_1\gamma_\mu(1-\gamma^5)\psi_2\}\{\bar{\psi}_3\gamma^\mu(1+\gamma^5)\psi_4\} \\ = & -\{\bar{\psi}_1(1+\gamma^5)\psi_4\}\{\bar{\psi}_3(1-\gamma^5)\psi_2\} + \{\bar{\psi}_1\gamma^5(1+\gamma^5)\psi_4\}\{\bar{\psi}_3\gamma^5(1-\gamma^5)\psi_2\} \\ & +\frac{1}{2}\{\bar{\psi}_1\gamma_\mu(1+\gamma^5)\psi_4\}\{\bar{\psi}_3\gamma^\mu(1-\gamma^5)\psi_2\} + \frac{1}{2}\{\bar{\psi}_1\gamma_\mu\gamma^5(1+\gamma^5)\psi_4\}\{\bar{\psi}_3\gamma^\mu\gamma^5(1-\gamma^5)\psi_2\} \\ = & -2\{\bar{\psi}_1(1+\gamma^5)\psi_4\}\{\bar{\psi}_3(1-\gamma^5)\psi_2\}\end{aligned}$$

Therefore

$$\begin{aligned}\{\bar{u}_b\gamma_\mu(1-\gamma^5)u_t\}\{\bar{u}_\nu\gamma^\mu(1-\gamma^5)v_{\bar{\ell}}\} &= -\{\bar{u}_b\gamma_\mu(1-\gamma^5)u_t\}\{\bar{v}_{\bar{\ell}}^C\gamma^\mu(1+\gamma^5)u_\nu^C\} \\ &= 2\{\bar{u}_b(1+\gamma^5)u_\nu^C\}\{\bar{v}_{\bar{\ell}}^C(1-\gamma^5)u_t\}\end{aligned}$$

Likewise

$$\{\bar{v}_{\bar{b}}\gamma_\mu(1-\gamma^5)v_{\bar{t}}\}\{\bar{v}_{\bar{\nu}}\gamma^\mu(1-\gamma^5)u_\ell\} = 2\{\bar{v}_{\bar{b}}(1+\gamma^5)v_{\bar{\nu}}^C\}\{\bar{u}_\ell^C(1-\gamma^5)v_{\bar{t}}\}$$

At top(anti-top) rest frame, suppose $\hat{\bar{\boldsymbol{\ell}}}$ $(\hat{\boldsymbol{\ell}})$ as unit vector of $\bar{\ell}$ (ℓ) flight direction

$$v_{\bar{\ell}} = N_{\bar{\ell}}\begin{pmatrix} \boldsymbol{\sigma}\cdot\hat{\bar{\boldsymbol{\ell}}}\phi_{\bar{\ell}} \\ \phi_{\bar{\ell}} \end{pmatrix} \qquad u_\ell = N_\ell\begin{pmatrix} \phi_\ell \\ \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\ell}}\phi_\ell \end{pmatrix}$$

$$
\begin{aligned}
\bar{v}_{\ell}^{C} &= -v_{\ell}^{T} C^{-1} = \left(\phi_{\ell}^{T} \boldsymbol{\sigma}^{*} \cdot \hat{\bar{\boldsymbol{\ell}}} \;\; \phi_{\ell}^{T} \right) \begin{bmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{bmatrix} \\
&= -\left(\phi_{\ell}^{T} i\sigma_2 \;\; \phi_{\ell}^{T} \boldsymbol{\sigma}^{*} \cdot \hat{\bar{\boldsymbol{\ell}}} i\sigma_2 \right) = \left((i\sigma_2 \phi_{\ell}^{*})^{\dagger} \;\; -(i\sigma_2 \phi_{\ell}^{*})^{\dagger} \boldsymbol{\sigma} \cdot \hat{\bar{\boldsymbol{\ell}}} \right)
\end{aligned}
$$

note that $\sigma_i^{*} \sigma_2 = -\sigma_2 \sigma_i$

$$
\begin{aligned}
\bar{u}_{\ell}^{C} &= -u_{\ell}^{T} C^{-1} = \left(\phi_{\ell}^{T} \;\; \phi_{\ell}^{T} \boldsymbol{\sigma}^{*} \cdot \hat{\boldsymbol{\ell}} \right) \begin{bmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{bmatrix} \\
&= -\left(\phi_{\ell}^{T} \boldsymbol{\sigma}^{*} \cdot \hat{\boldsymbol{\ell}} i\sigma_2 \;\; \phi_{\ell}^{T} i\sigma_2 \right) = \left(-(i\sigma_2 \phi_{\ell}^{*})^{\dagger} \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\ell}} \;\; (i\sigma_2 \phi_{\ell}^{*})^{\dagger} \right)
\end{aligned}
$$

At top rest frame

$$
\begin{aligned}
& \{\bar{u}_b \gamma_\mu (1-\gamma^5) u_t\} \{\bar{u}_\nu \gamma^\mu (1-\gamma^5) v_{\bar{\ell}}\} = 2\{\bar{u}_b (1+\gamma^5) u_\nu^C\} \{\bar{v}_{\bar{\ell}}^{C} (1-\gamma^5) u_t\} \\
= \;& 2\{\bar{u}_b (1+\gamma^5) u_\nu^C\} \left\{ \left((i\sigma_2 \phi_{\bar{\ell}}^{*})^{\dagger} \;\; -(i\sigma_2 \phi_{\bar{\ell}}^{*})^{\dagger} \boldsymbol{\sigma} \cdot \hat{\bar{\boldsymbol{\ell}}} \right) (1-\gamma^5) \begin{pmatrix} \phi_t \\ 0 \end{pmatrix} \right\} \\
= \;& 2\{\bar{u}_b (1+\gamma^5) u_\nu^C\} \left\{ (i\sigma_2 \phi_{\bar{\ell}}^{*})^{\dagger} \left(1 + \boldsymbol{\sigma} \cdot \hat{\bar{\boldsymbol{\ell}}} \right) \phi_t \right\}
\end{aligned}
$$

$$
\begin{aligned}
& \{\bar{v}_{\bar{b}} \gamma_\mu (1-\gamma^5) v_{\bar{t}}\} \{\bar{v}_{\bar{\nu}} \gamma^\mu (1-\gamma^5) u_{\ell}\} = 2\{\bar{v}_{\bar{b}} (1+\gamma^5) v_{\bar{\nu}}^C\} \{\bar{u}_{\ell}^{C} (1-\gamma^5) v_{\bar{t}}\} \\
= \;& 2\{\bar{v}_{\bar{b}} (1+\gamma^5) v_{\bar{\nu}}^C\} \left\{ \left(-(i\sigma_2 \phi_{\ell}^{*})^{\dagger} \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\ell}} \;\; (i\sigma_2 \phi_{\ell}^{*})^{\dagger} \right) (1-\gamma^5) \begin{pmatrix} 0 \\ \phi_{\bar{t}} \end{pmatrix} \right\} \\
= \;& 2\{\bar{u}_b (1+\gamma^5) u_\nu^C\} \left\{ (i\sigma_2 \phi_{\ell}^{*})^{\dagger} \left(1 + \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\ell}} \right) \phi_{\bar{t}} \right\}
\end{aligned}
$$

If you take the quantizing axis of top along with the lepton flight direction, i.e.

$$
1 + \boldsymbol{\sigma} \cdot \hat{\bar{\boldsymbol{\ell}}} = 1 + \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\ell}} = 1 + \sigma_3 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}
$$

Thus only

$$
\phi_t = \phi_{\bar{t}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
$$

allowed. This means top is polarized to $\bar{\ell}$ flight direction and anti-top is polarized to opposite direction of ℓ flight direction.

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