How to make random numbers yield a given distribution from uniform random numbers

r : uniform random number in interval $[0,1]$

$f(x)$: given probability density function

$$f: x \in [-\infty, \infty] \rightarrow f(x) \in [0, \infty] \qquad \int_{-\infty}^{\infty} f(x)\,dx = 1 \tag{1}$$

We define the following two monotone nondecreasing functions $F(x), G(x)$

$$\begin{aligned} F(x) &\equiv \int_{-\infty}^{x} f(x)dx & F(-\infty) = 0, F(\infty) = 1 \\ G(x) &\equiv F^{-1}(x) & G(0) = -\infty, G(0) = \infty \end{aligned} \tag{2}$$

Using this function G, map of uniform random numbers

$$y = G(r) \tag{3}$$

yields density function $f(x)$.

(Proof)

We suppose $g(x)$ as the probability density function which y comes from, then

$$g(y)dy = p(r)dr \qquad y = G(r) \tag{4}$$

and

$$\begin{aligned} g(y)dy &= g(y)\left|\frac{\partial y}{\partial r}\right|dr \\ &= g(y)G'(F(y))dr \end{aligned} \tag{5}$$

By definition $p(r) = 1$, therefore

$$g(y) = \frac{1}{G'(F(y))} \tag{6}$$

Here we remind that function G is inverse function of function F

$$G'(F(y)) \cdot F'(y) = 1 \tag{7}$$

After all, we conclude

$$g(y) = F'(y) = f(y) \tag{8}$$

which means y comes from a density distribution f.

How to make random numbers yield Gaussian distribution

If indefinite integral of given distribution is not holomorphic, the method mentioned above can't be suitable. In case of Gaussian, the following method is known.

r_1, r_2 : uniformly distributed random numbers in interval $[0,1]$ respectively.

We define the following map:

$$\begin{aligned} &(r_1, r_2) \in \boldsymbol{D}: [0,1] \times [0,1] \rightarrow (x_1, x_2) \in \Re^2 \\ &x_1 = g(r_2)\cos(2\pi r_1) \\ &x_2 = g(r_2)\sin(2\pi r_1) \qquad \text{where } g(r) = \sqrt{-2\log r} \end{aligned} \tag{9}$$

Variables x_1 and x_2 come from a probability density function:

$$f(x) = N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{10}$$

[Proof]

We suppose that $p(r_1, r_2)$ and $\varphi(x_1, x_2)$ are probability density functions which (r_1, r_2) and (x_1, x_2) come from respectively, and then

$$\begin{aligned} p(r_1, r_2) d r_1 d r_2 &= \varphi(x_1, x_2) d x_1 d x_2 \\ &= \varphi(x_1, x_2) \left| \frac{\partial(x_1, x_2)}{\partial(r_1, r_2)} \right| d r_1 d r_2 \end{aligned} \tag{11}$$

By definition,

$$p(r_1, r_2) \equiv 1 \tag{12}$$

$$\left| \frac{\partial(x_1, x_2)}{\partial(r_1, r_2)} \right| = \begin{vmatrix} \frac{\partial x_1}{\partial r_1} & \frac{\partial x_2}{\partial r_1} \\ \frac{\partial x_1}{\partial r_2} & \frac{\partial x_2}{\partial r_2} \end{vmatrix} = -2\pi g(r_2) g'(r_2) \tag{13}$$

Therefore,

$$\varphi(x_1, x_2) = -\frac{1}{2\pi\, g(r_2) g'(r_2)} \tag{14}$$

Here we apply the definition $g(r) = \sqrt{-2\log r}$ in Eq.(9)

$$\begin{aligned} g'(r) &= -\frac{1}{\sqrt{-2\log r}} \cdot \frac{1}{r} \\ &= -\frac{1}{r\, g(r)} \end{aligned} \tag{15}$$

$$\varphi(x_1, x_2) = \frac{1}{2\pi} r_2 = \frac{1}{2\pi} g^{-1}\left(\sqrt{x_1^2 + x_2^2} \right) \tag{16}$$

Finally,

$$\begin{aligned} \varphi(x_1) &= \int_{-\infty}^{\infty} \varphi(x_1, x_2) d x_1 d x_2 \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{x_1^2 + x_2^2}{2}} d x_1 d x_2 \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}} \end{aligned} \tag{17}$$

[Q.E.D]