Yuji Takeuchi, Oct. 22, 2007

Definition of the spin spin correlation coefficient $\boldsymbol{\kappa}$

1. Define a quantization basis for t and $\bar{t}$: $\hat{\boldsymbol{\zeta}}$
   - off-diagonal, helicity, beam axis (z-axis), ...
2. Quantize t and $\bar{t}$ spins along $\hat{\boldsymbol{\zeta}}$: $s_t(\hat{\boldsymbol{\zeta}})$, $s_{\bar{t}}(\hat{\boldsymbol{\zeta}})$
   - spin1/2 $\Rightarrow s_t, s_{\bar{t}} = \uparrow, \downarrow$
3. Number of $t\bar{t}$ in state $s_t s_{\bar{t}}$: $N_{s_t s_{\bar{t}}}$
4. $\kappa(\hat{\boldsymbol{\zeta}}) = \dfrac{N_{\uparrow\uparrow} + N_{\downarrow\downarrow} - N_{\uparrow\downarrow} - N_{\downarrow\uparrow}}{N_{\uparrow\uparrow} + N_{\downarrow\downarrow} + N_{\uparrow\downarrow} + N_{\downarrow\uparrow}}$
   - Depends on the quantization basis

Angular distribution of $\boldsymbol{\ell}^{\pm}$ in $\boldsymbol{t\bar{t}}$ with $\boldsymbol{\kappa}$

Suppose a spin state of $t\bar{t}$ in the basis of $\hat{\boldsymbol{\zeta}}$ at the production as

$$\begin{aligned} \langle t, \bar{t}\,| &= \langle i|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \langle i|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + \langle i|\downarrow\uparrow\rangle\langle\downarrow\uparrow| + \langle i|\downarrow\downarrow\rangle\langle\downarrow\downarrow| \\ &\equiv \alpha_i\langle\uparrow\uparrow| + \beta_i\langle\uparrow\downarrow| + \gamma_i\langle\downarrow\uparrow| + \delta_i\langle\downarrow\downarrow| \end{aligned},$$

where i represents index of initial states, and

$$N_{\uparrow\uparrow} \propto \sum_i |\alpha_i|^2,\; N_{\uparrow\downarrow} \propto \sum_i |\beta_i|^2,\; N_{\downarrow\uparrow} \propto \sum_i |\gamma_i|^2,\; N_{\downarrow\downarrow} \propto \sum_i |\delta_i|^2.$$

Suppose $\ell^{\pm}$ flight directions in $t(\bar{t}\,)$ rest frame are $(\theta_{\pm},\, \varphi_{\pm})$ in the basis of $\hat{\boldsymbol{\zeta}}$.

Since $t(\bar{t}\,)$ spin is polarized along $\ell^{\pm}$ flight direction, we can write $t(\bar{t}\,)$ spin state after decay as

$$|t(\ell)\rangle = e^{-i\varphi_+/2}\cos\frac{\theta_+}{2}|\uparrow\rangle + e^{i\varphi_+/2}\sin\frac{\theta_+}{2}|\downarrow\rangle \equiv a|\uparrow\rangle + b|\downarrow\rangle$$

$$\big|\bar{t}\,(\bar{\ell}\,)\big\rangle = -\,e^{i\varphi_-/2}\sin\frac{\theta_-}{2}|\uparrow\rangle + e^{-i\varphi_-/2}\cos\frac{\theta_-}{2}|\downarrow\rangle \equiv c|\uparrow\rangle + d|\downarrow\rangle.$$

i.e. a spin state of $t\bar{t}$ after decay is

$$|t, \bar{t}\,\rangle = ac|\uparrow\uparrow\rangle + ad|\uparrow\downarrow\rangle + bc|\downarrow\uparrow\rangle + bd|\downarrow\downarrow\rangle,$$

where

$$ac = -e^{-i(\varphi_+ - \varphi_-)/2}\cos\frac{\theta_+}{2}\sin\frac{\theta_-}{2}$$

$$ad = e^{-i(\varphi_+ + \varphi_-)/2}\cos\frac{\theta_+}{2}\cos\frac{\theta_-}{2}$$

$$bc = e^{i(\varphi_+ + \varphi_-)/2}\sin\frac{\theta_+}{2}\sin\frac{\theta_-}{2}$$

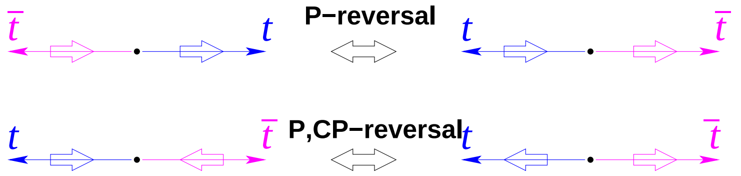
$$bd = -e^{i(\varphi_+ - \varphi_-)/2}\sin\frac{\theta_+}{2}\cos\frac{\theta_-}{2}.$$

The matrix element of $t\bar{t}$ decaying dileptons with $(\theta_{\pm},\, \varphi_{\pm})$ is

$$
\begin{aligned}
|\mathfrak{M}|^2 &\propto \sum_i \left[|\alpha_i ac + \beta_i ad + \gamma_i bc + \delta_i bd|^2 \right] \\
&= \sum_i \; (\alpha_i^* a^* c^* + \beta_i^* a^* d^* + \gamma_i^* b^* c^* + \delta_i^* b^* d^*)(\alpha_i ac + \beta_i ad + \gamma_i bc + \delta_i bd) \\
&= \sum_i \; \left[\, |\alpha_i ac|^2 + |\beta_i ad|^2 + |\gamma_i bc|^2 + |\delta_i bd|^2 + (\text{interference terms}) \,\right]
\end{aligned}
$$

Interference terms have the factor $e^{\pm i\varphi_+}$ or $e^{\pm i\varphi_-}$ and are vanished if integrated over $\varphi_\pm$, i.e.

$$
\begin{aligned}
\int |\mathfrak{M}|^2 d\varphi &\propto \sum_i \left[\, |\alpha_i ac|^2 + |\beta_i ad|^2 + |\gamma_i bc|^2 + |\delta_i bd|^2 \right] \\
&= \cos^2\frac{\theta_+}{2}\sin^2\frac{\theta_-}{2} \sum_i |\alpha_i|^2 + \sin^2\frac{\theta_+}{2}\cos^2\frac{\theta_-}{2} \sum_i |\delta_i|^2 \\
&\quad + \cos^2\frac{\theta_+}{2}\cos^2\frac{\theta_-}{2} \sum_i |\beta_i|^2 + \sin^2\frac{\theta_+}{2}\sin^2\frac{\theta_-}{2} \sum_i |\gamma_i|^2 \\
&\propto \frac{(1+\cos\theta_+)(1-\cos\theta_-)}{4} N_{\uparrow\uparrow} + \frac{(1-\cos\theta_+)(1+\cos\theta_-)}{4} N_{\downarrow\downarrow} \\
&\quad + \frac{(1+\cos\theta_+)(1+\cos\theta_-)}{4} N_{\uparrow\downarrow} + \frac{(1-\cos\theta_+)(1-\cos\theta_-)}{4} N_{\downarrow\uparrow} \\
&= \frac{1-\cos\theta_+\cos\theta_-}{2}(N_{\uparrow\uparrow} + N_{\downarrow\downarrow}) + \frac{\cos\theta_+ - \cos\theta_-}{4}(N_{\uparrow\uparrow} - N_{\downarrow\downarrow}) \\
&\quad + \frac{1+\cos\theta_+\cos\theta_-}{2}(N_{\uparrow\downarrow} + N_{\downarrow\uparrow}) + \frac{\cos\theta_+ + \cos\theta_-}{4}(N_{\uparrow\downarrow} - N_{\downarrow\uparrow})
\end{aligned}
$$



Assume P conserved: $N_{\uparrow\uparrow} = N_{\downarrow\downarrow}$, $N_{\uparrow\downarrow} = N_{\downarrow\uparrow}$

Assume CP conserved: $N_{\uparrow\downarrow} = N_{\downarrow\uparrow}$

Here we suppose $N_P \equiv N_{\uparrow\uparrow} = N_{\downarrow\downarrow}$, and $N_A \equiv N_{\uparrow\downarrow} = N_{\downarrow\uparrow}$, i.e. $\kappa = (N_P - N_A)/(N_P + N_A)$.

$$
\begin{aligned}
\int |\mathfrak{M}|^2 d\varphi &\propto (1-\cos\theta_+\cos\theta_-)N_P + (1+\cos\theta_+\cos\theta_-)N_A \\
&= (N_P + N_A)(1-\kappa\cos\theta_+\cos\theta_-)
\end{aligned}
$$

Finally we obtain

$$
\frac{1}{\sigma}\frac{d^2\sigma}{d\cos\theta_+ d\cos\theta_-} = \frac{1-\kappa\cos\theta_+\cos\theta_-}{4}.
$$

Here we wiped off the interference terms by integrated over $\varphi_\pm$, and assumed $N_{\uparrow\uparrow} = N_{\downarrow\downarrow}$, $N_{\uparrow\downarrow} = N_{\downarrow\uparrow}$.

*) In Review of Particle Physics 2006, p.522, it is written as

$$
\frac{1}{\sigma}\frac{d^2\sigma}{d\cos\theta_+ d\cos\theta_-} = \frac{1+\kappa\cos\theta_+\cos\theta_-}{4}.
$$

This is because spin quantization basis is taken $\hat{\zeta}$ for top, while $-\hat{\zeta}$ for anti-top.