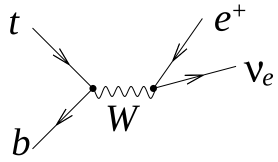
Top Quark Decay

4 Fermi V-A interaction



Matrix element of top quark decay $t \to \ell^+ \nu_\ell b$ is

$$\begin{aligned}\mathfrak{M} &= \varepsilon_\mu^* \,\bar{u}_b \left(-i\frac{g}{\sqrt{2}} V_{tb} \gamma^\mu \frac{1-\gamma^5}{2} \right) u_t \times \varepsilon_\nu \,\bar{u}_u \left(-i\frac{g}{\sqrt{2}} \gamma^\nu \frac{1-\gamma^5}{2} \right) v_d \\ &\propto \varepsilon_\mu^* \,\bar{u}_b \gamma^\mu (1-\gamma^5) u_t \times \varepsilon_\nu \,\bar{u}_u \gamma^\nu (1-\gamma^5) v_d \end{aligned} ,$$

where u_u and v_d represent spinors of up-type(ν_ℓ) and down-type(ℓ^+) quark from W, respectively.

Considering $\sum \varepsilon^\mu \varepsilon^{\nu *} = -g^{\mu\nu} + p_W^\mu p_W^\nu / m_W^2$, we get

$$\mathfrak{M} \propto \bar{u}_b \gamma_\mu (1-\gamma^5) u_t (-g^{\mu\nu} + p_W^\mu p_W^\nu / m_W^2) \,\bar{u}_u \gamma_\nu (1-\gamma^5) v_d \ ,$$

Here $p_W = p_u + p_d$. Then

$$\begin{aligned} p_W^\nu \bar{u}_u \gamma_\nu (1-\gamma^5) v_d &= (p_u^\nu + p_d^\nu) \bar{u}_u \gamma_\nu (1-\gamma^5) v_d \\ &= \bar{u}_u \not{p}_u (1-\gamma^5) v_d + \bar{u}_u \not{p}_d (1-\gamma^5) v_d \\ &= m_u \bar{u}_u (1-\gamma^5) v_d - m_v \bar{u}_u (1+\gamma^5) v_d = 0 \end{aligned} ,$$

where we assume the masses of up-type and down-type quarks from W are negligibly small, i.e. $m_u = m_d = 0$.

Thus, we omit the term includes $p_W^\mu p_W^\nu$, and get

$$\mathfrak{M} \propto \bar{u}_b \gamma_\mu (1-\gamma^5) u_t \,\bar{u}_u \gamma^\nu (1-\gamma^5) v_d$$

In general, we have the following equation:

$$\begin{aligned} \bar{\psi} \gamma^\mu (1-\gamma^5) \phi &= \{\bar{\psi} \gamma^\mu (1-\gamma^5) \phi\}^T \\ &= -\phi^T \{\gamma^\mu (1-\gamma^5)\}^T \bar{\psi}^T \\ &= \bar{\phi}^C C \{\gamma^\mu (1-\gamma^5)\}^T C^{-1} \psi^C = -\bar{\phi}^C (1-\gamma^5) \gamma^\mu \psi^C \\ &= -\bar{\phi}^C \gamma^\mu (1+\gamma^5) \psi^C \ , \end{aligned}$$

where $\psi^C = C \bar{\psi}^T = i\gamma^2 \psi^*$ and $C = i\gamma^2 \gamma^0 = \begin{bmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{bmatrix}$.

Using Fierz transformation, we obtain the following equation as well:

$$\begin{aligned} &\{\bar{\psi}_1 \gamma_\mu (1-\gamma^5) \psi_2\} \{\bar{\psi}_3 \gamma^\mu (1+\gamma^5) \psi_4\} \\ &\qquad = -\{\bar{\psi}_1 (1+\gamma^5) \psi_4\} \{\bar{\psi}_3 (1-\gamma^5) \psi_2\} \\ &\qquad + \frac{1}{2} \{\bar{\psi}_1 \gamma_\mu (1+\gamma^5) \psi_4\} \{\bar{\psi}_3 \gamma^\mu (1-\gamma^5) \psi_2\} \\ &\qquad + \frac{1}{2} \{\bar{\psi}_1 \gamma_\mu \gamma^5 (1+\gamma^5) \psi_4\} \{\bar{\psi}_3 \gamma^\mu \gamma^5 (1-\gamma^5) \psi_2\} \\ &\qquad + \{\bar{\psi}_1 \gamma^5 (1+\gamma^5) \psi_4\} \{\bar{\psi}_3 \gamma^5 (1-\gamma^5) \psi_2\} \\ &\qquad = -2 \{\bar{\psi}_1 (1+\gamma^5) \psi_4\} \{\bar{\psi}_3 (1-\gamma^5) \psi_2\} \end{aligned}$$

Applying the two equations above to the invariant matrix element for the top quark decay,

$$\begin{aligned} \mathfrak{M} &\propto \bar{u}_b \gamma_\mu (1-\gamma^5) u_t \,\bar{u}_u \gamma^\nu (1-\gamma^5) v_d \\ &= -\{\bar{u}_b \gamma_\mu (1-\gamma^5) u_t\} \{\bar{v}_d^C \gamma^\mu (1+\gamma^5) u_u^C\} \\ &= 2 \{\bar{u}_b (1+\gamma^5) u_u^C\} \{\bar{v}_d^C (1-\gamma^5) u_t\} \equiv 2 \mathfrak{M}_{b\nu} \mathfrak{M}_{t\ell} \end{aligned}$$

$$
\begin{aligned}
|\mathfrak{M}|^2 \propto & \sum_{s_d}\sum_{s_u,s_b} |\mathfrak{M}_{t\ell}|^2 |\bar{u}_b(1+\gamma^5)u_u^C|^2 \\
= & \sum_{s_d} |\mathfrak{M}_{t\ell}|^2 \sum_{s_u,s_b} \bar{u}_b(1+\gamma^5)u_u^C\bar{u}_u^C(1-\gamma^5)u_b \\
= & \sum_{s_d} |\mathfrak{M}_{t\ell}|^2 \sum_{s_u,s_b} \mathrm{Tr}\{(1+\gamma^5)u_u^C\bar{u}_u^C(1-\gamma^5)u_b\bar{u}_b\} \\
= & \sum_{s_d} |\mathfrak{M}_{t\ell}|^2 \mathrm{Tr}\Big\{(1+\gamma^5)\Big(\gamma^2\not{p}_u^*\gamma^2\Big)(1-\gamma^5)\not{p}_b\Big\} \\
= & -\sum_{s_d} |\mathfrak{M}_{t\ell}|^2 \mathrm{Tr}\Big\{(1+\gamma^5)\not{p}_u(1-\gamma^5)\not{p}_b\Big\} \\
= & -2\sum_{s_d} |\mathfrak{M}_{t\ell}|^2 \mathrm{Tr}\Big\{\not{p}_u\not{p}_b+\gamma^5\not{p}_u\not{p}_b\Big\} \\
= & -8\sum_{s_d} |\mathfrak{M}_{t\ell}|^2 (p_u\cdot p_b)
\end{aligned}
$$

Next, we consider the factor $\mathfrak{M}_{t\ell}$. In the top rest frame, define spinors of top quark and down-type qaurk as

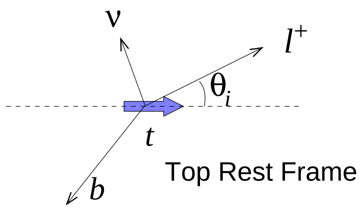
$$
u_t=\sqrt{2m_t}\begin{pmatrix}\chi_t\\0\end{pmatrix} \qquad v_d=\sqrt{E_d}\begin{pmatrix}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{d}}\phi_d\\ \phi_d\end{pmatrix},
$$

respectively, where χ_t and ϕ_d are two component spinors of top and down-type quark, $\hat{\boldsymbol{d}}$ and E_d are unit vector of the flight direction of down-type quark and its energy in the top rest frame.

$$
v_d^C=i\gamma^2 v_d^*\propto\begin{bmatrix}0 & i\sigma_2\\ -i\sigma_2 & 0\end{bmatrix}\begin{pmatrix}\boldsymbol{\sigma}^*\cdot\hat{\boldsymbol{d}}\phi_d^*\\ \phi_d^*\end{pmatrix}=\begin{pmatrix}i\sigma_2\phi_d^*\\ -i\sigma_2\boldsymbol{\sigma}^*\cdot\hat{\boldsymbol{d}}\phi_d^*\end{pmatrix}=\begin{pmatrix}i\sigma_2\phi_d^*\\ \boldsymbol{\sigma}\cdot\hat{\boldsymbol{d}}i\sigma_2\phi_d^*\end{pmatrix}
$$

$$
\begin{aligned}
\mathfrak{M}_{t\ell} = & \sqrt{2m_tE_d}\{\bar{v}_d^C(1-\gamma^5)u_t\} \\
= & \sqrt{2m_tE_d}\big(-\phi_d^T i\sigma_2 \;\; \phi_d^T i\sigma_2\boldsymbol{\sigma}\cdot\hat{\boldsymbol{d}}\big)\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{pmatrix}\chi_t\\0\end{pmatrix} \\
= & -\sqrt{2m_tE_d}\,\phi_d^T i\sigma_2\big(1+\boldsymbol{\sigma}\cdot\hat{\boldsymbol{d}}\big)\chi_t
\end{aligned}
$$

Assume the decay occurs in z-x plane, and the top quark is polarized to z-direction. Also define the angles of the flight direction of down-type quark w.r.t. z-axis as θ_d.



$$
\chi_t=\begin{pmatrix}1\\0\end{pmatrix}
$$

$$
\begin{aligned}
1+\boldsymbol{\sigma}\cdot\hat{\boldsymbol{d}} = & 1+\sigma_3\cos\theta_d+\sigma_1\sin\theta_d=\begin{bmatrix}1+\cos\theta_d & \sin\theta_d\\ \sin\theta_d & 1-\cos\theta_d\end{bmatrix} \\
= & 2\begin{bmatrix}\cos^2\frac{\theta_d}{2} & \sin\frac{\theta_d}{2}\cos\frac{\theta_d}{2}\\ \sin\frac{\theta_d}{2}\cos\frac{\theta_d}{2} & \sin^2\frac{\theta_d}{2}\end{bmatrix}
\end{aligned}
$$

This means $\mathfrak{M}_{t\ell}=0$ for

$$
\phi_d^T i\sigma_2=\begin{pmatrix}-\sin\frac{\theta_d}{2} & \cos\frac{\theta_d}{2}\end{pmatrix}
$$

Conversely only allowed state is

$$\phi_d^T i\sigma_2 = \begin{pmatrix} \cos\frac{\theta_d}{2} & \sin\frac{\theta_d}{2} \end{pmatrix}$$

i.e.

$$\phi_d^T = \begin{pmatrix} \cos\frac{\theta_d}{2} & \sin\frac{\theta_d}{2} \end{pmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{pmatrix} \cos\frac{\theta_d}{2} & \sin\frac{\theta_d}{2} \end{pmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{pmatrix} \sin\frac{\theta_d}{2} & -\cos\frac{\theta_d}{2} \end{pmatrix}$$

which means ℓ^+ has positive helicity. Then, finally

$$\begin{aligned}
\mathfrak{M}_{t\ell} &= -\sqrt{2m_t E_d}\,\phi_d^T i\sigma_2 \big(1 + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{d}}\big)\chi_t \\
&= -2\sqrt{2m_t E_d} \begin{pmatrix} \cos\frac{\theta_d}{2} & \sin\frac{\theta_d}{2} \end{pmatrix} \begin{bmatrix} \cos^2\frac{\theta_d}{2} & \sin\frac{\theta_d}{2}\cos\frac{\theta_d}{2} \\ \sin\frac{\theta_d}{2}\cos\frac{\theta_d}{2} & \sin^2\frac{\theta_d}{2} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= -2\sqrt{2m_t E_d}\cos\frac{\theta_d}{2} \\
|\mathfrak{M}|^2 &\propto \sum_{s_d} |\mathfrak{M}_{t\ell}|^2 (p_u \cdot p_b) \propto m_t E_d (1 - \cos\theta_d)(p_u \cdot p_b)
\end{aligned}$$

In the top quark rest frame $p_t = (m_t, \vec{0})$

$$\begin{aligned}
|\mathfrak{M}|^2 &\propto m_t E_d (1 - \cos\theta_d)(p_u \cdot p_b) \\
&= ((m_t, m_t\hat{\boldsymbol{s}}) \cdot (E_d, \vec{p}_d))(p_u \cdot p_b) \\
&= \{(p_t + m_t s_t) \cdot p_d\}(p_u \cdot p_b)
\end{aligned}$$

where $s_t = (0,\ \hat{\boldsymbol{s}})$ in the top quark rest frame, and $\hat{\boldsymbol{s}}$ is unit vector parallel to top quark polarization.