

Acceleration in Spherical Coordinates

$$a_r = \ddot{r} - r\dot{\theta}^2 \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r} \frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right)$$

Miscellaneous Formulas

$$\begin{aligned} \vec{F} &= -\nabla U & \dot{\vec{p}} &= \vec{F} \\ \vec{l} &= \vec{r} \times \vec{p} & \text{Angular momentum} \\ \vec{N} &= \vec{r} \times \vec{F} & \text{Moment of force} & \quad \dot{\vec{l}} = \vec{N} \\ \mu &= \frac{mM}{m+M} & \text{Reduced mass} \end{aligned}$$

Rotating Coordinate System

Consider S' that rotates about an axis in S by angular velocity $\vec{\omega}$

$$\begin{aligned} \frac{d\vec{A}}{dt} &= \frac{d'\vec{A}}{dt} + \vec{\omega} \times \vec{A} & \frac{d\vec{r}}{dt} &= \vec{\omega} \times \vec{r} \\ \frac{d^2\vec{A}}{dt^2} &= \frac{d'^2\vec{A}}{dt^2} + 2\vec{\omega} \times \frac{d'\vec{A}}{dt} + \vec{\omega} \times (\vec{\omega} \times \vec{A}) + \dot{\vec{\omega}} \times \vec{A} & \dot{\vec{\omega}} &= \frac{d\vec{\omega}}{dt} = \frac{d'\vec{\omega}}{dt} \end{aligned}$$

Moment of Inertia

$$I = \int \rho r^2 dv \quad \text{Moment of inertia about Line } \ell \quad r: \text{Distance from Line } \ell$$

Inertia Tensor

$$\begin{aligned} I_{xx} &= \int \rho(y^2 + z^2) dv & I_{yy} &= \int \rho(z^2 + x^2) dv & I_{zz} &= \int \rho(x^2 + y^2) dv \\ I_{xy} &= -\int \rho xy dv & I_{yz} &= -\int \rho yz dv & I_{zx} &= -\int \rho zx dv \\ L_i &= I_{ij} \omega_j & I_{ij} &= I_{ji} \end{aligned}$$

Moment of inertia about a line through origin O with direction cosine (λ, μ, ν) .

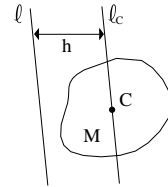
$$\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu} \quad \lambda^2 + \mu^2 + \nu^2 = 1$$

$$\begin{aligned} I &= I_{xx} \lambda^2 + I_{yy} \mu^2 + I_{zz} \nu^2 + 2 I_{xy} \lambda \mu + 2 I_{yz} \mu \nu + 2 I_{zx} \nu \lambda \\ &= I_{ij} \lambda_i \lambda_j \end{aligned} \quad \text{where } \lambda_i = \lambda, \mu, \nu$$

Moment of inertia about a parallel axis

ℓ_C : A straight line through center of gravity C parallel to line ℓ .
 I : Moment of inertia about line ℓ
 I_C : Moment of inertia about line ℓ_C

$$I = I_C + Mh^2$$

**Principal Moment of Inertia**

$$\begin{aligned} \vec{\rho} &= \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} & \vec{r} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \text{Coordinate transformation } \vec{\rho} &= U^{-1} \vec{r} & U^T &= U^{-1} \\ \vec{\lambda} &= \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} & \text{Direction cosine vector } \vec{\lambda}' &= U^{-1} \vec{\lambda} \end{aligned}$$

$$I = \vec{\lambda}^T \{I_{ij}\} \vec{\lambda} = \vec{\lambda}'^T U^T \{I_{ij}\} U \vec{\lambda}' = \vec{\lambda}'^T \begin{pmatrix} A & & \\ & B & \\ & & C \end{pmatrix} \vec{\lambda}'$$

Examples of principal moment of inertia about simple shapes

$$A = I_{xx} \quad B = I_{yy} \quad C = I_{zz}$$

Spherical shell $A = B = C = \frac{2}{3}Ma^2$ a : Radius

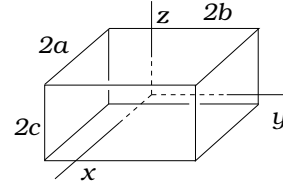
Sphere $A = B = C = \frac{2}{5}Ma^2$

Rectangular pole

$$A = \frac{1}{3}M(b^2 + c^2)$$

$$B = \frac{1}{3}M(c^2 + a^2)$$

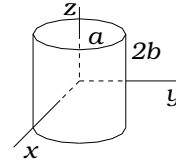
$$C = \frac{1}{3}M(a^2 + b^2)$$



Circular cylinder

$$A = B = M\left(\frac{a^2}{4} + \frac{b^2}{3}\right)$$

$$C = \frac{1}{2}Ma^2$$

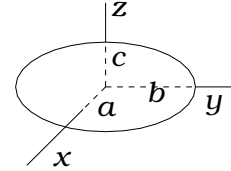


Ellipsoidal

$$A = \frac{1}{5}M(b^2 + c^2)$$

$$B = \frac{1}{5}M(c^2 + a^2)$$

$$C = \frac{1}{5}M(a^2 + b^2)$$



Kinetic Energy of Rigid Body

$$K = \frac{1}{2}\vec{\omega} \cdot \vec{L} = \frac{1}{2}I\omega^2 \quad I : \text{Moment of inertia about the direction of } \vec{\omega}$$

Motion of Rigid Body around Fixed Point

Euler Equations

$$A\dot{\omega}_x - (B - C)\omega_y\omega_z = N_x \quad A = I_{xx}$$

$$B\dot{\omega}_y - (C - A)\omega_z\omega_x = N_y \quad B = I_{yy} \quad \text{Principal moment of inertia}$$

$$C\dot{\omega}_z - (A - B)\omega_x\omega_y = N_z \quad C = I_{zz}$$

$$\frac{dK}{dt} = \vec{\omega} \cdot \vec{N} : \text{Energy equation} \quad K : \text{Kinetic energy}$$

$$\vec{L} \cdot \frac{d\vec{L}}{dt} = \frac{1}{2} \frac{d\vec{L}^2}{dt} = \frac{1}{2} \frac{dL^2}{dt} = \vec{N} \cdot \vec{L}$$

Euler Angles

OXYZ : Space fixed frame $\Rightarrow \vec{R}$

Oxyz : Body fixed frame $\Rightarrow \vec{r}$

$$\vec{r} = \begin{pmatrix} \cos\varphi \cos\theta \cos\psi - \sin\varphi \sin\psi & \sin\varphi \cos\theta \cos\psi + \cos\varphi \sin\psi & -\sin\theta \cos\psi \\ -\cos\varphi \cos\theta \sin\psi - \sin\varphi \cos\psi & -\sin\varphi \cos\theta \sin\psi + \cos\varphi \cos\psi & \sin\theta \sin\psi \\ \cos\varphi \sin\theta & \sin\varphi \sin\theta & \cos\theta \end{pmatrix} \vec{R}$$

$$\omega_x = \dot{\theta} \sin\psi - \dot{\varphi} \sin\theta \cos\psi$$

$$\dot{\theta} = \omega_x \sin\psi + \omega_y \cos\psi$$

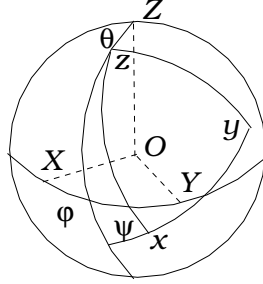
$$\omega_y = \dot{\theta} \cos\psi + \dot{\varphi} \sin\theta \sin\psi$$

$$\iff$$

$$\dot{\varphi} = \operatorname{cosec}\theta(-\omega_x \cos\psi + \omega_y \sin\psi)$$

$$\omega_z = \dot{\varphi} \cos\theta + \dot{\psi}$$

$$\dot{\psi} = \omega_z - \cot\theta(-\omega_x \cos\psi + \omega_y \sin\psi)$$



$$\vec{R} \rightarrow \vec{r}: R_z(-\psi) \circ R_y(-\theta) \circ R_z(-\varphi)$$

Analytical Mechanics of System of Particles

$$L = T - U : \text{Lagrangian}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) = \frac{\partial L}{\partial q_r} \quad (r = 1, \dots, f)$$

$$L = T - q\varphi + q\vec{v} \cdot \vec{A} : \text{Lagrangian of charged particle in electric and magnetic field}$$

Hamiltonian

$$p_r = \frac{\partial L}{\partial \dot{q}_r}$$

$$H(p_r, q_r) = \sum_{r=1}^f p_r \dot{q}_r - L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t)$$

$$\dot{q}_r = \frac{\partial H}{\partial p_r} \quad \dot{p}_r = -\frac{\partial H}{\partial q_r} : \text{Canonical Equations}$$

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\varphi : \text{Hamiltonian of charged particle in EM field}$$

$$H = \frac{1}{2m} \left(p_r^2 + \frac{1}{r^2} p_\theta^2 + \frac{1}{r^2 \sin^2\theta} p_\varphi^2 \right) + V(r) : \text{Hamiltonian in spherical coordinate}$$

Canonical Transformation

$$H(p, q, t) \rightarrow \bar{H}(P, Q, t)$$

Consider generating functions like $W = W(q_1, \dots, q_f, Q_1, \dots, Q_f, t) \rightarrow W(q, Q, t)$

Generating functions (+ sign for variables q and P , - sign for variables Q and p)

$$W(q, Q) \quad p = \frac{\partial W}{\partial q} \quad P = -\frac{\partial W}{\partial Q}$$

$$W(q, P) \quad p = \frac{\partial W}{\partial q} \quad Q = \frac{\partial W}{\partial P}$$

$$W(Q, p) \quad P = -\frac{\partial W}{\partial Q} \quad q = -\frac{\partial W}{\partial p}$$

$$W(p, P) \quad Q = \frac{\partial W}{\partial P} \quad q = -\frac{\partial W}{\partial p}$$

$$\bar{H} = H + \frac{\partial W}{\partial t}$$

Hamilton-Jacobi Partial Differential Equation

- i. Make Hamiltonian. $H(q, p, t)$
- ii. Solve H-J Equation: $\frac{\partial}{\partial t}W + H\left(q, \frac{\partial W}{\partial q}, t\right) = 0$

Complete solution of H-J eq. will be $W(q, \alpha, t)$ where $\alpha = (\alpha_1, \dots, \alpha_f)$ is constant.

$$\text{iii.} \quad p_r = \frac{\partial W}{\partial q_r} \quad \beta_r = \frac{\partial W}{\partial \alpha_r} \quad \alpha, \beta : \text{Constants}$$

Case that H doesn't contain t explicitly

- i. Solve H-J equation: $H\left(q, \frac{\partial S}{\partial q}\right) = E$

In this case, separation of variables is usually applicable and E and S are

$$E = E(\alpha_1, \dots, \alpha_f) \quad S = S(q_1, \dots, q_f, \alpha_1, \dots, \alpha_f)$$

$E = \alpha_1 + \dots + \alpha_f$ usually, or $E = \alpha_1$ in case to obtain trajectory.

$$\text{ii.} \quad p_r = \frac{\partial S}{\partial q_r} \quad \beta_r = -\frac{\partial E}{\partial \alpha_r}t + \frac{\partial S}{\partial \alpha_r}$$

Poisson Bracket

$$\{u, v\} = \sum_r \left(\frac{\partial u}{\partial q_r} \frac{\partial v}{\partial p_r} - \frac{\partial u}{\partial p_r} \frac{\partial v}{\partial q_r} \right)$$

$$\{u, v\} = -\{v, u\} \quad \{u, u\} = 0 \quad \{u, vw\} = v\{u, w\} + \{u, v\}w$$

$$\{u_1 + u_2, v_1 + v_2\} = \{u_1, v_1\} + \{u_2, v_1\} + \{u_1, v_2\} + \{u_2, v_2\}$$

$$\{u, v, w\} \equiv \{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\} = 0 \quad (\text{Jacobi identity})$$

$$\dot{F} = \{F, H\} + \frac{\partial F}{\partial t} \quad \dot{q} = \{q, H\} \quad \dot{p} = \{p, H\}$$

Relativistic Mechanics

$$x^\mu = (ct, \vec{x}) \quad u^\mu = \frac{dx^\mu}{d\tau} = (c\gamma, \gamma\vec{v}) \quad \frac{dt}{d\tau} = \gamma$$

$$p^\mu = m_0 u^\mu = \left(\frac{E}{c}, \vec{p} \right) \quad F^\mu = \frac{dp^\mu}{d\tau} = \left(\gamma \frac{\vec{F} \cdot \vec{v}}{c}, \gamma \vec{F} \right)$$

$$\partial_\mu = \left(\frac{\partial}{\partial x^0}, \nabla \right) = \left(\frac{\partial}{\partial ct}, \nabla \right)$$

$$k^\mu = \left(\frac{\omega}{c}, \vec{k} \right) \quad j^\mu = (c\rho, \vec{j}) \quad A^\mu = \left(\frac{\phi}{c}, \vec{A} \right)$$

$$L = m_0 c^2 \left(1 - \sqrt{1 - \beta^2} \right) - U \quad H = mc^2 - m_0 c^2 + U$$

Charged Particle

$$L = m_0 c^2 \left(1 - \sqrt{1 - \beta^2} \right) - q\varphi + q\vec{v} \cdot \vec{A} \quad H = \sqrt{m_0^2 c^4 + c^2 (\vec{p} - q\vec{A})^2} - m_0 c^2 + q\varphi$$

$$pc(\text{GeV}) = 0.3 \cdot \rho(\text{m}) \cdot B(\text{Tesla})$$