Maxwell Equation

$$\nabla \cdot \boldsymbol{D} = \rho \qquad \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \qquad \nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{J}$$

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} = \varepsilon_0 (1 + \chi_e) \boldsymbol{E} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} = \varepsilon_r \varepsilon_0 \boldsymbol{E}$$

$$\boldsymbol{B} = \mu \boldsymbol{H} = \mu_0 (1 + \chi_m) \boldsymbol{H} = \mu_0 \boldsymbol{H} + \boldsymbol{M} = \mu_r \mu_0 \boldsymbol{H}$$

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}$$

Lorentz Force

Electromagnetic Potential

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad \boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla \phi$$

Lorentz Gauge

$$\Box \boldsymbol{A} = \mu_0 \boldsymbol{J} \qquad \Box \phi = \frac{1}{\varepsilon_0} \rho \qquad \nabla \cdot \boldsymbol{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$$

Potential energy of a charged particle in electromagnetic field

$$U = q(\phi - \boldsymbol{v} \cdot \boldsymbol{A})$$

Point Charge

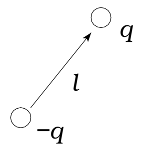
$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \frac{\boldsymbol{r}}{r} \qquad \phi = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r}$$

Electric Dipole

$$\boldsymbol{p} = q\boldsymbol{l}$$

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0} \left(-\frac{\boldsymbol{p}}{r^3} + \frac{3(\boldsymbol{p} \cdot \boldsymbol{r})\boldsymbol{r}}{r^5} \right)$$

$$\phi = \frac{1}{4\pi\varepsilon_0} \frac{\boldsymbol{p} \cdot \boldsymbol{r}}{r^3}$$

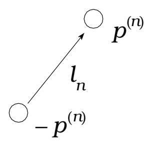


Multipole

$p^{(n)}$: Multipole of order 2^n

$$p^{(n+1)} \equiv (n+1) p^{(n)} l_n \qquad l_n \to 0,\ p^{(n)} \to \infty$$

$$p^{(0)} \equiv q$$

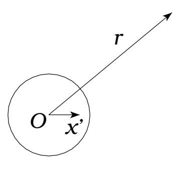


$$V_n = \frac{(-1)^n}{4\pi\varepsilon_0} \frac{p^{(n)}}{n!} \frac{\partial^n}{\partial l_0 \partial l_1 \cdots \partial l_{n-1}} \left(\frac{1}{r} \right) = \frac{p^{(n)}}{4\pi\varepsilon_0} \frac{Y_n(\theta, \varphi)}{r^{n+1}}$$

$$Y_n(\theta, \varphi) = \sum_{m=0}^{n} (a_{nm} \cos m\varphi + b_{nm} \sin m\varphi) P_n^m(\cos\theta)$$

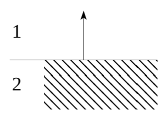
$$\frac{1}{|\boldsymbol{r} - \boldsymbol{x}'|} = \frac{1}{r} \sum_{l=0}^{\infty} \left(\frac{x'}{r} \right)^l P_l(\cos\theta')$$

θ' : angle between $\boldsymbol{r}$ and $\boldsymbol{x}'$



Boundary Condition

$$\begin{cases} E_{1\parallel} = E_{2\parallel} \\ D_{1\perp} - D_{2\perp} = \sigma \end{cases} \quad \text{or} \quad \begin{cases} \phi_1 = \phi_2 \\ \varepsilon_1 \frac{\partial \phi_1}{\partial n} - \varepsilon_2 \frac{\partial \phi_2}{\partial n} = -\sigma \end{cases}$$



Solution of Laplace Equation

Spherical Coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}=0$$

$$\phi=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(a_n r^n+\frac{b_n}{r^{n+1}}\right)(c_m\cos m\varphi+d_m\sin m\varphi)P_n^m(\cos\theta)$$

Axial symmetry around z-axis

$$\phi=\sum_{m=0}^{\infty}\left(a_n r^n+\frac{b_n}{r^{n+1}}\right)P_n(\cos\theta)$$

Cylindrical Coordinates (uniform along z-axis)

$$r\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{\partial^2\phi}{\partial\theta^2}=0$$

$$\phi=\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\left(c_n r^n+\frac{d_n}{r^2}\right)+(a_0\theta+b_0)(c_0\log r+d_0)$$

Magnetic Dipole

$$U=-\boldsymbol{m}\cdot\boldsymbol{H}\qquad\qquad \boldsymbol{F}=\boldsymbol{\nabla}(\boldsymbol{m}\cdot\boldsymbol{H})$$

$$N_\theta=\frac{\partial}{\partial\theta}(\boldsymbol{m}\cdot\boldsymbol{H})\qquad [\boldsymbol{N}=\boldsymbol{m}\times\boldsymbol{H}]$$

$$\boldsymbol{H}=\frac{1}{4\pi\mu_0}\left(-\frac{\boldsymbol{m}}{r^3}+\frac{3(\boldsymbol{m}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5}\right)$$

Degaussing Coefficient

$$\boldsymbol{H}=\boldsymbol{H}_0-\boldsymbol{H}'=\boldsymbol{H}_0-\nu\boldsymbol{M}=\boldsymbol{H}_0-(D/\mu_0)\boldsymbol{M}$$

where $\boldsymbol{H}'$: degaussing force $\quad\nu$: degaussing fraction $\quad D$: degaussing coefficient

Sphere $\quad D=1/3$

Ellipsoid

i. $\quad a>b=c \qquad e=\sqrt{a^2-b^2}/a$

$$\nu_a=\frac{1-e^2}{\mu_0 e^2}\left(\frac{1}{2e}\log\frac{1+e}{1-e}-1\right)\qquad \nu_b=\nu_c=\frac{1}{2\mu_0 e^2}\left(1-\frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right)$$

ii. $\quad a=b>c \qquad e=\sqrt{a^2-c^2}/a$

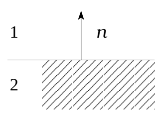
$$\nu_a=\nu_b=\frac{\sqrt{1-e^2}}{2\mu_0 e^2}\left(\frac{1}{e}\sin^{-1}e-\sqrt{1-e^2}\right)\qquad \nu_c=\frac{1}{\mu_0 e^2}\left(1-\frac{\sqrt{1-e^2}}{e}\sin^{-1}e\right)$$

Long Cylinder $\quad D=1/2$

Boundary Condition in Magnetic Field

Surface current $\boldsymbol{j}$ on boundary surface

$$(\boldsymbol{H}_2-\boldsymbol{H}_1)\times\boldsymbol{n}\equiv\boldsymbol{j}$$
$$\boldsymbol{B}_{1\perp}=\boldsymbol{B}_{2\perp}$$



Biot-savart's Law

$$d\boldsymbol{H}=\frac{I}{4\pi}\frac{d\boldsymbol{s}\times\boldsymbol{r}}{r^3}$$

Vector Potential

$$\boldsymbol{A}=\frac{\mu}{4\pi}\int\frac{\boldsymbol{J}}{r}d\boldsymbol{v}$$

Magnetic flux

$$\phi=\iint\boldsymbol{B}\cdot d\boldsymbol{s}=\iint\boldsymbol{\nabla}\times\boldsymbol{A}\cdot d\boldsymbol{s}=\oint\boldsymbol{A}\cdot d\boldsymbol{t}$$

Multipole Expansion

$$\boldsymbol{A}(\boldsymbol{r})=\frac{\mu}{4\pi}\int\frac{\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{r}-\boldsymbol{x}'|}d\,v'=\frac{\mu}{4\pi}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int\boldsymbol{J}(\boldsymbol{x}')|\boldsymbol{x}'|^{l}\mathrm{P}_l(\cos\theta')d\,v'$$

$$\boldsymbol{A}_0(\boldsymbol{r})=\frac{\mu}{4\pi}\frac{1}{r}\int\boldsymbol{J}(\boldsymbol{x}')d\,v'\equiv 0$$

$$\begin{aligned}\boldsymbol{A}_1(\boldsymbol{r})&=\frac{\mu}{4\pi}\frac{1}{r^3}\int\boldsymbol{J}(\boldsymbol{x}')(\boldsymbol{x}'\cdot\boldsymbol{r})d\,v'\\&=\frac{\mu}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}\end{aligned}\qquad\qquad \boldsymbol{m}\equiv\frac{1}{2}\int\boldsymbol{x}'\times\boldsymbol{J}(\boldsymbol{x}')d\,v'$$

Boundary Condition

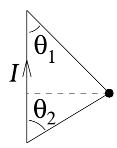
$$\begin{cases}\boldsymbol{A}_1=\boldsymbol{A}_2\\ \left(\frac{1}{\mu_2}\boldsymbol{\nabla}\times\boldsymbol{A}_2-\frac{1}{\mu_1}\boldsymbol{\nabla}\times\boldsymbol{A}_1\right)\times\boldsymbol{n}=\boldsymbol{j}\end{cases}$$

Magnetic Field by Current

- Straight current with infinite length

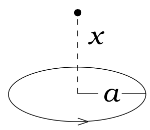
$$H=\frac{I}{2\pi a^2}r\ (r\le a)\qquad\qquad H=\frac{I}{2\pi r}\ (r\ge a)$$

- Straight current with finite length



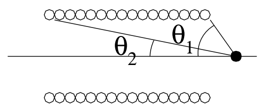
$$H=\frac{I}{4\pi r}(\cos\theta_1+\cos\theta_2)$$

- Circular current



$$H=\frac{a^2 I}{2(a^2+x^2)^{3/2}}$$

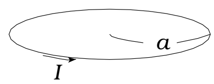
- Solenoid with finite length



$$H=\frac{n\,I}{2}(\cos\theta_2-\cos\theta_1)$$

Vector Potential by Circular Current

$$A_\phi=\frac{I}{4\pi}\int_0^{2\pi}\frac{a\cos(\varphi-\varphi')d\varphi'}{[r^2+r'^2-2r\,r'\cos(\varphi-\varphi')+(z-z')^2]^{1/2}}$$



Energy in Magnetic Field by Current

$$U=\frac{1}{2}\int\boldsymbol{J}\cdot\boldsymbol{A}d\,v$$

Poynting Vector

$$\boldsymbol{S}\equiv\boldsymbol{E}\times\boldsymbol{H}\qquad\qquad\frac{1}{\sqrt{\varepsilon_0\mu_0}}=c\qquad\qquad\frac{H}{E}=\sqrt{\frac{\varepsilon}{\mu}}$$

Solution of $\Box\phi=\frac{\rho}{\varepsilon_0}$ and $\Box\boldsymbol{A}=\mu_0\boldsymbol{J}$

$$\phi(\boldsymbol{x},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\boldsymbol{x}',t-r/c)}{r}d\boldsymbol{x}'\qquad\qquad r\equiv|\boldsymbol{x}-\boldsymbol{x}'|$$

$$\boldsymbol{A}(\boldsymbol{x},t)=\frac{\mu_0}{4\pi}\int\frac{\boldsymbol{J}(\boldsymbol{x}',t-r/c)}{r}\,d\boldsymbol{x}'$$

Liénard-Wiechert Potentials

$$\phi(\boldsymbol{x},t)=\frac{q}{4\pi\varepsilon_0}\frac{1}{s}\qquad \boldsymbol{A}(\boldsymbol{x},t)=\frac{\mu_0 q}{4\pi}\frac{\boldsymbol{v}}{s}\qquad s\equiv r-\frac{\boldsymbol{r}\cdot\boldsymbol{v}}{c}$$

$$\boldsymbol{r}\equiv\boldsymbol{x}-\boldsymbol{x}'(t')\qquad r\equiv|\boldsymbol{x}-\boldsymbol{x}'(t')|\qquad \boldsymbol{v}\equiv\dot{\boldsymbol{x}}'(t')\qquad t'=t-r/c$$

$$\boldsymbol{E}=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{s^3}\left(\boldsymbol{r}-\frac{r\boldsymbol{v}}{c}\right)\left(1-\frac{v^2}{c^2}\right)+\frac{1}{c^2s^3}\left[\boldsymbol{r}\times\left\{\left(\boldsymbol{r}-\frac{r\boldsymbol{v}}{c}\right)\times\dot{\boldsymbol{v}}\right\}\right]\right)$$

$$\boldsymbol{B}=\frac{\mu_0 q}{4\pi}\left(\frac{\boldsymbol{v}\times\boldsymbol{r}}{s^3}\left(1-\frac{v^2}{c^2}\right)+\frac{1}{c\,s^3}\frac{\boldsymbol{r}}{r}\times\left[\boldsymbol{r}\times\left\{\left(\boldsymbol{r}-\frac{r\boldsymbol{v}}{c}\right)\times\dot{\boldsymbol{v}}\right\}\right]\right)=\frac{\boldsymbol{r}}{c\,r}\times\boldsymbol{E}$$

If $v\ll c$, then $s\simeq r$

$$\boldsymbol{E}=\frac{q}{4\pi\varepsilon_0c^2r^3}\boldsymbol{r}\times(\boldsymbol{r}\times\dot{\boldsymbol{v}})\qquad \boldsymbol{S}=\frac{q^2\boldsymbol{r}}{16\pi^2\varepsilon_0c^3r^7}\{\boldsymbol{r}\times(\boldsymbol{r}\times\dot{\boldsymbol{v}})\}^2$$

$$\begin{aligned}\frac{dW}{dt}&=\frac{q^2}{16\pi^2\varepsilon_0c^3}\int d\Omega\{\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\dot{\boldsymbol{v}})\}^2\\&=\frac{q^2}{6\pi\varepsilon_0c^3}(\dot{\boldsymbol{v}}(t))^2\end{aligned}\qquad \hat{\boldsymbol{n}}\equiv\frac{\boldsymbol{r}}{r}$$

Radio Scattering by Point Charge

E_0 : Amplitude of incident electromagnetic wave

$$\frac{d\sigma}{d\Omega}=\frac{1}{(4\pi\varepsilon_0)^2}\frac{q^2}{c^4E_0^2}|\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\dot{\boldsymbol{v}})|^2\qquad \hat{\boldsymbol{n}}\equiv\frac{\boldsymbol{r}}{r}$$

i. Thomson scattering (scattering by free electron)

$$d\sigma_T=\left(\frac{e^2}{4\pi\varepsilon_0m\,c^2}\right)^2\frac{1+\cos^2\theta}{2}d\Omega$$

$$\sigma_T=\frac{8\pi}{3}\left(\frac{e^2}{4\pi\varepsilon_0m\,c^2}\right)^2=\frac{8\pi}{3}a_0^2\qquad a_0\equiv\frac{e^2}{4\pi\varepsilon_0m\,c^2}\ :\ \text{Classical electron radius}$$

ii. Rayleigh scattering (scattering by electron bound in atom)

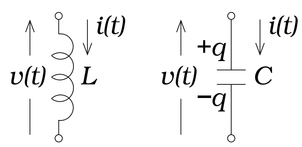
$$d\sigma_R=d\sigma_T\frac{\omega^4}{(\omega_0^2-\omega^2)^2}\qquad \omega_0\ :\ \text{Eigenfrequency}$$

Electric circuit

Alternating Current($V=V_0\,e^{i\omega t}$)

$$v(t)=L\,\frac{di(t)}{dt}\qquad Z=i\omega L$$

$$i(t)=C\,\frac{dv(t)}{dt}\qquad Z=\frac{1}{i\omega C}$$



RMS value

$$V=V_0\,e^{i\omega t}\qquad I=I_0\,e^{i(\omega t-\phi)}\qquad \tilde{V}=V_0/\sqrt{2}\qquad \tilde{I}=I_0/\sqrt{2}$$

$$\tilde{P}=\frac{1}{2}\mathrm{Re}(VI^*)=\frac{1}{2}V_0\,I_0\cos\phi=\tilde{V}\tilde{I}\,\cos\phi\qquad \cos\phi\ :\ \text{phase factor}$$