Maxwell Equation

$$\nabla\cdot\boldsymbol{D}=\rho$$
$$\nabla\times\boldsymbol{E}+\frac{\partial\boldsymbol{B}}{\partial t}=0$$
$$\nabla\cdot\boldsymbol{B}=0$$
$$\nabla\times\boldsymbol{H}-\frac{\partial\boldsymbol{D}}{\partial t}=\boldsymbol{j}$$

$$\begin{aligned}\boldsymbol{D}=\varepsilon\boldsymbol{E} &=\varepsilon_0(1+\chi_e)\boldsymbol{E}\\ &=\varepsilon_0\boldsymbol{E}+\boldsymbol{P}\\ &=\varepsilon_r\varepsilon_0\boldsymbol{E}\end{aligned}$$

$$\begin{aligned}\boldsymbol{B}=\mu\boldsymbol{H} &=\mu_0(1+\chi_m)\boldsymbol{H}\\ &=\mu_0\boldsymbol{H}+\mu_0\boldsymbol{M}\\ &=\mu_r\mu_0\boldsymbol{H}\end{aligned}$$

$$\boldsymbol{P}=\chi_e\varepsilon_0\boldsymbol{E}$$
$$\boldsymbol{M}=\chi_m\boldsymbol{H}$$

$$\boldsymbol{F}=q\boldsymbol{E}+q\boldsymbol{v}\times\boldsymbol{B}$$
Lorentz Force

Electromagnetic Potential

$$\boldsymbol{B}=\nabla\times\boldsymbol{A}\qquad \boldsymbol{E}=-\frac{\partial\boldsymbol{A}}{\partial t}-\nabla\phi\qquad \text{Lorenz gauge}$$

$$\Box\boldsymbol{A}=\mu_0\boldsymbol{j}\qquad \Box\phi=\frac{1}{\varepsilon_0}\rho\qquad \nabla\cdot\boldsymbol{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}=0$$

Potential energy of a charged particle in electromagnetic field

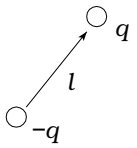
$$U=q(\phi-\boldsymbol{v}\cdot\boldsymbol{A})$$

Point Charge

$$\boldsymbol{E}=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\frac{\boldsymbol{r}}{r}\qquad \phi=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

Electric Dipole

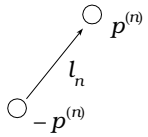
$$\boldsymbol{p}=q\boldsymbol{l}$$
$$\boldsymbol{E}=\frac{1}{4\pi\varepsilon_0}\left\{-\frac{\boldsymbol{p}}{r^3}+\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5}\right\}$$
$$\phi=\frac{1}{4\pi\varepsilon_0}\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3}$$



Multipole

$p^{(n)}$: Multipole of order 2^n

$$p^{(n+1)}\equiv(n+1)p^{(n)}l_n\qquad l_n\to 0,\ p^{(n)}\to\infty$$
$$p^{(0)}\equiv q$$

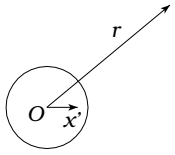


$$V_n=\frac{(-1)^n}{4\pi\varepsilon_0}\frac{p^{(n)}}{n!}\frac{\partial^n}{\partial l_0\partial l_1\cdots\partial l_{n-1}}\left(\frac{1}{r}\right)=\frac{p^{(n)}}{4\pi\varepsilon_0}\frac{Y_n(\theta,\varphi)}{r^{n+1}}$$

$$Y_n(\theta,\varphi)=\sum_{m=0}^{n}(a_{nm}\cos m\varphi+b_{nm}\sin m\varphi)P_n^m(\cos\theta)$$

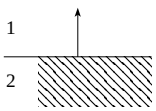
$$\frac{1}{|\boldsymbol{r}-\boldsymbol{x}'|}=\frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{x'}{r}\right)^l P_l(\cos\theta')$$

θ' : angle between $\boldsymbol{r}$ and $\boldsymbol{x'}$



Boundary Condition

$$\begin{cases}E_{1\parallel}=E_{2\parallel}\\ D_{1\perp}-D_{2\perp}=\sigma\end{cases}\qquad\text{or}\qquad\begin{cases}\phi_1=\phi_2\\ \varepsilon_1\frac{\partial\phi_1}{\partial n}-\varepsilon_2\frac{\partial\phi_2}{\partial n}=-\sigma\end{cases}$$



Solution of Laplace Equation

Spherical Coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}=0$$

$$\phi=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(a_nr^n+\frac{b_n}{r^{n+1}}\right)(c_m\cos m\varphi+d_m\sin m\varphi)P_n^m(\cos\theta)$$

Axial symmetry around z-axis

$$\phi=\sum_{m=0}^{\infty}\left(a_nr^n+\frac{b_n}{r^{n+1}}\right)P_n(\cos\theta)$$

Cylindrical Coordinates (uniform along z-axis)

$$r\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{\partial^2\phi}{\partial\theta^2}=0$$

$$\phi=\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\left(c_nr^n+\frac{d_n}{r^2}\right)+(a_0\theta+b_0)(c_0\log r+d_0)$$

Magnetic Dipole

$$U=-\boldsymbol{m}\cdot\boldsymbol{H}\qquad\qquad\boldsymbol{F}=\boldsymbol{\nabla}(\boldsymbol{m}\cdot\boldsymbol{H})$$

$$N_\theta=\frac{\partial}{\partial\theta}(\boldsymbol{m}\cdot\boldsymbol{H})\qquad[\boldsymbol{N}=\boldsymbol{m}\times\boldsymbol{H}]$$

$$\boldsymbol{H}=\frac{1}{4\pi\mu_0}\left\{-\frac{\boldsymbol{m}}{r^3}+\frac{3(\boldsymbol{m}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5}\right\}$$

Degaussing Coefficient

$$\boldsymbol{H}=\boldsymbol{H}_0-\boldsymbol{H}_d=\boldsymbol{H}_0-\nu\boldsymbol{M}$$

where $\boldsymbol{H}_d$: demagnetizing field strength ν : demagnetizing factor

Sphere $\nu=1/3$

Ellipsoid

i. $a>b=c\qquad e=\sqrt{a^2-b^2}/a$

$$\nu_a=\frac{1-e^2}{e^2}\left(\frac{1}{2e}\log\frac{1+e}{1-e}-1\right)\qquad\nu_b=\nu_c=\frac{1}{2e^2}\left(1-\frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right)$$

ii. $a=b>c\qquad e=\sqrt{a^2-c^2}/a$

$$\nu_a=\nu_b=\frac{\sqrt{1-e^2}}{2e^2}\left(\frac{1}{e}\sin^{-1}e-\sqrt{1-e^2}\right)\qquad\nu_c=\frac{1}{e^2}\left(1-\frac{\sqrt{1-e^2}}{e}\sin^{-1}e\right)$$

Long Cylinder $\nu=1/2$ (along the cylinder) $\nu=0$ (transverse to the cylinder)

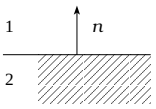
Ampere's Law

$$\oint_{\partial S}\boldsymbol{H}\cdot d\boldsymbol{t}=\int_S\boldsymbol{j}\cdot d\boldsymbol{s}$$

Boundary Condition in Magnetic Field

Surface current $\boldsymbol{j}$ on boundary surface

$$(\boldsymbol{H}_2-\boldsymbol{H}_1)\times\boldsymbol{n}=\boldsymbol{j}$$
$$\boldsymbol{B}_{1\perp}=\boldsymbol{B}_{2\perp}$$



Biot-savart's Law

$$d\boldsymbol{H} = \frac{I}{4\pi}\frac{d\boldsymbol{t}\times\boldsymbol{r}}{r^3}$$

Vector Potential

$$\boldsymbol{A} = \frac{\mu}{4\pi}\int\frac{\boldsymbol{j}}{r}dv$$

Magnetic flux

$$\phi = \int_S \boldsymbol{B}\cdot d\boldsymbol{s} = \int_S \boldsymbol{\nabla}\times\boldsymbol{A}\cdot d\boldsymbol{s} = \oint_{\partial S}\boldsymbol{A}\cdot d\boldsymbol{t}$$

Multipole Expansion

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi}\int\frac{\boldsymbol{j}(\boldsymbol{x}')}{|\boldsymbol{r}-\boldsymbol{x}'|}dv' = \frac{\mu}{4\pi}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int\boldsymbol{j}(\boldsymbol{x}')|\boldsymbol{x}'|^l P_l(\cos\theta')dv'$$

$$\boldsymbol{A}_0(\boldsymbol{r}) = \frac{\mu}{4\pi}\frac{1}{r}\int\boldsymbol{j}(\boldsymbol{x}')dv' \equiv 0$$

$$\begin{aligned}\boldsymbol{A}_1(\boldsymbol{r}) &= \frac{\mu}{4\pi}\frac{1}{r^3}\int\boldsymbol{j}(\boldsymbol{x}')(\boldsymbol{x}'\cdot\boldsymbol{r})dv' \\ &= \frac{\mu}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}\end{aligned} \qquad \boldsymbol{m} \equiv \frac{1}{2}\int\boldsymbol{x}'\times\boldsymbol{j}(\boldsymbol{x}')dv'$$

Boundary Condition

$$\begin{cases}\boldsymbol{A}_1 = \boldsymbol{A}_2 \\ \left(\frac{1}{\mu_2}\boldsymbol{\nabla}\times\boldsymbol{A}_2 - \frac{1}{\mu_1}\boldsymbol{\nabla}\times\boldsymbol{A}_1\right)\times\boldsymbol{n} = \boldsymbol{j}\end{cases}$$

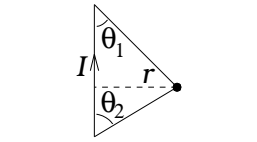
Magnetic Field by Current

- Infinite-length straight current uniformly within radius a

$$H = \frac{I}{2\pi a^2}r\ (r\le a) \qquad H = \frac{I}{2\pi r}\ (r\ge a)$$

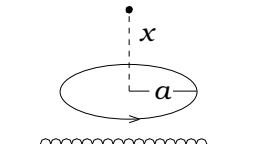
- Straight current with finite length

$$H = \frac{I}{4\pi r}(\cos\theta_1 + \cos\theta_2)$$



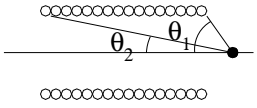
- Circular current with radius a

$$H = \frac{a^2 I}{2(a^2+x^2)^{3/2}}$$



- Solenoid with finite length with n turns per unit length

$$H = \frac{nI}{2}(\cos\theta_2 - \cos\theta_1)$$



Vector Potential by Circular Current

$$A_\phi = \frac{I}{4\pi}\int_0^{2\pi}\frac{a\cos(\varphi-\varphi')d\varphi'}{[r^2+r'^2-2r\,r'\cos(\varphi-\varphi')+(z-z')^2]^{1/2}}$$

Energy in Magnetic Field by Current

$$U = \frac{1}{2}\int\boldsymbol{j}\cdot\boldsymbol{A}dv$$

Poynting Vector

$$\boldsymbol{S} \equiv \boldsymbol{E}\times\boldsymbol{H} \qquad \frac{1}{\sqrt{\varepsilon_0\mu_0}} = c \qquad \frac{H}{E} = \sqrt{\frac{\varepsilon}{\mu}}$$

$$u = \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B})$$

Solution of $\Box\phi = \dfrac{\rho}{\varepsilon_0}$ and $\Box\boldsymbol{A} = \mu_0\boldsymbol{j}$

$$\phi(\boldsymbol{x},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{x}', t - r/c)}{r}\, d\boldsymbol{x}' \qquad r \equiv |\boldsymbol{x} - \boldsymbol{x}'|$$

$$\boldsymbol{A}(\boldsymbol{x},t) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{j}(\boldsymbol{x}', t - r/c)}{r}\, d\boldsymbol{x}'$$

Liénard-Wiechert Potentials

$$\phi(\boldsymbol{x},t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{s} \qquad \boldsymbol{A}(\boldsymbol{x},t) = \frac{\mu_0 q}{4\pi}\frac{\boldsymbol{v}}{s} \qquad s \equiv r - \frac{\boldsymbol{r}\cdot\boldsymbol{v}}{c}$$

$$\boldsymbol{r} \equiv \boldsymbol{x} - \boldsymbol{x}'(t') \qquad r \equiv |\boldsymbol{x} - \boldsymbol{x}'(t')| \qquad \boldsymbol{v} \equiv \dot{\boldsymbol{x}}'(t') \qquad t' = t - r/c$$

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{s^3}\left(\boldsymbol{r} - \frac{r\boldsymbol{v}}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \frac{1}{c^2 s^3}\boldsymbol{r}\times\left\{\left(\boldsymbol{r} - \frac{r\boldsymbol{v}}{c}\right)\times\dot{\boldsymbol{v}}\right\}\right]$$

$$\boldsymbol{B} = \frac{\mu_0 q}{4\pi}\left[\frac{\boldsymbol{v}\times\boldsymbol{r}}{s^3}\left(1 - \frac{v^2}{c^2}\right) + \frac{1}{c\,s^3}\frac{\boldsymbol{r}}{r}\times\left[\boldsymbol{r}\times\left\{\left(\boldsymbol{r} - \frac{r\boldsymbol{v}}{c}\right)\times\dot{\boldsymbol{v}}\right\}\right]\right] = \frac{\boldsymbol{r}}{c\,r}\times\boldsymbol{E}$$

If $v \ll c$, then $s \simeq r$

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 c^2 r^3}\boldsymbol{r}\times(\boldsymbol{r}\times\dot{\boldsymbol{v}}) \qquad \boldsymbol{S} = \frac{q^2\boldsymbol{r}}{16\pi^2\varepsilon_0 c^3 r^7}\{\boldsymbol{r}\times(\boldsymbol{r}\times\dot{\boldsymbol{v}})\}^2$$

$$\frac{dW}{dt} = \frac{q^2}{16\pi^2\varepsilon_0 c^3}\int d\Omega\{\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\dot{\boldsymbol{v}})\}^2 = \frac{q^2}{6\pi\varepsilon_0 c^3}|\dot{\boldsymbol{v}}(t)|^2 \qquad \hat{\boldsymbol{n}} \equiv \frac{\boldsymbol{r}}{r}$$

Radio Scattering by Point Charge

E_0 : Amplitude of incident electromagnetic wave

$$\frac{d\sigma}{d\Omega} = \frac{1}{(4\pi\varepsilon_0)^2}\frac{q^2}{c^4 E_0^2}|\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\dot{\boldsymbol{v}})|^2 \qquad \hat{\boldsymbol{n}} \equiv \frac{\boldsymbol{r}}{r}$$

i. Thomson scattering (scattering by free electron)

$$d\sigma_T = \left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2\frac{1+\cos^2\theta}{2}d\Omega$$

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 = \frac{8\pi}{3}a_0^2 \qquad a_0 \equiv \frac{e^2}{4\pi\varepsilon_0 mc^2} : \text{ Classical electron radius}$$

ii. Rayleigh scattering (scattering by electron bound in atom)

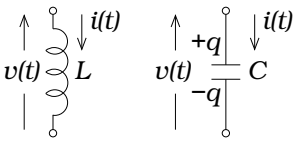
$$d\sigma_R = d\sigma_T\frac{\omega^4}{(\omega_0^2 - \omega^2)^2} \qquad \omega_0 : \text{Eigenfrequency}$$

Electric circuit

Alternating Current($V = V_0\, e^{i\omega t}$)

$$v(t) = L\frac{di(t)}{dt} \qquad Z = i\omega L$$

$$i(t) = C\frac{dv(t)}{dt} \qquad Z = \frac{1}{i\omega C}$$



RMS value

$$V = V_0\, e^{i\omega t} \qquad I = I_0\, e^{i(\omega t - \phi)} \qquad \tilde{V} = V_0/\sqrt{2} \qquad \tilde{I} = I_0/\sqrt{2}$$

$$\tilde{P} = \frac{1}{2}\mathrm{Re}(VI^*) = \frac{1}{2}V_0 I_0\cos\phi = \tilde{V}\tilde{I}\cos\phi \qquad \cos\phi : \text{phase factor}$$