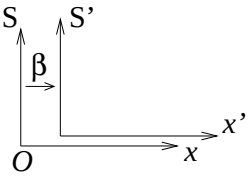
Lorentz Transformation

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \cosh\varphi & -\sinh\varphi \\ -\sinh\varphi & \cosh\varphi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\varphi \qquad \gamma^2\beta^2 = \gamma^2 - 1 = \sinh^2\varphi$$

$$\gamma\beta = \sinh\varphi \ (\varphi > 0) \qquad \beta = \tanh\varphi \qquad \varphi\text{: Rapidity}$$



β/γ of Particle

$$\vec{\beta} = \frac{\vec{p}}{E} \qquad \gamma = \frac{E}{m} \qquad \vec{\beta}\gamma = \frac{\vec{p}}{m}$$

$$\alpha \equiv \frac{\beta\gamma}{\gamma+1} = \tanh\left(\frac{\varphi}{2}\right) \qquad \alpha^2 = \frac{\gamma-1}{\gamma+1} \qquad 1-\alpha^2 = \frac{2}{\gamma+1} \qquad 1+\alpha^2 = \frac{2\gamma}{\gamma+1}$$

$$\gamma = \frac{1+\alpha^2}{1-\alpha^2} \qquad \gamma+1 = \frac{2}{1-\alpha^2} \qquad \beta = \frac{2\alpha}{1+\alpha^2} \qquad \gamma(1+\beta) = \frac{1+\alpha}{1-\alpha}$$

General Lorentz transformation $S \to S'$

$$a^\mu = (a^0; \vec{a}) \text{ in } S \qquad a'^\mu = (a'^0; \vec{a}') \text{ in } S' \qquad p^\mu = (p^0; \vec{p}) \text{ : 4-mom. for } S \to S'$$

$$a'^0 = (p^0 a^0 - \vec{p}\cdot\vec{a})/m = (a^0 - \vec{\beta}\cdot\vec{a})\gamma \qquad \text{where } m = \sqrt{p^2}$$

$$\vec{p}\cdot\vec{a} = p^0 a^0 - m a'^0 \qquad \vec{\beta}\cdot\vec{a} = a^0 - \tfrac{1}{\gamma} a'^0$$

$$\vec{a}' = \vec{a} + \frac{1}{m}\left(\frac{\vec{p}\cdot\vec{a}}{m+p^0} - a^0\right)\vec{p} = \vec{a} + \left(\frac{\gamma}{1+\gamma}\vec{\beta}\cdot\vec{a} - a^0\right)\vec{\beta}\gamma$$

$$= \vec{a} - (a'^0 + a^0)\frac{\vec{p}}{m+p^0} = \vec{a} - (a'^0 + a^0)\frac{\vec{\beta}\gamma}{1+\gamma}$$

Boost $S' \to S$

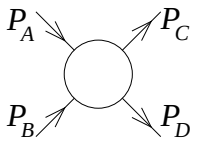
$$a^0 = (p^0 a'^0 + \vec{p}\cdot\vec{a}')/m = (a'^0 + \vec{\beta}\cdot\vec{a}')\gamma$$

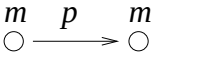
$$\vec{a} = \vec{a}' + \frac{1}{m}\left(\frac{\vec{p}\cdot\vec{a}'}{m+p^0} + a'^0\right)\vec{p} = \vec{a}' + \left(\frac{\gamma}{1+\gamma}\vec{\beta}\cdot\vec{a}' + a'^0\right)\vec{\beta}\gamma$$

$$= \vec{a}' + (a^0 + a'^0)\frac{\vec{p}}{m+p^0} = \vec{a}' + (a^0 + a'^0)\frac{\vec{\beta}\gamma}{1+\gamma}$$

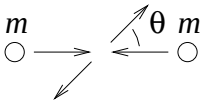
Invariant Variables (Mandelstam variables)

$$s = (P_\mathrm{A} + P_\mathrm{B})^2 = (P_\mathrm{C} + P_\mathrm{D})^2$$
$$t = (P_\mathrm{A} - P_\mathrm{C})^2 = (P_\mathrm{B} - P_\mathrm{D})^2$$
$$u = (P_\mathrm{A} - P_\mathrm{D})^2 = (P_\mathrm{B} - P_\mathrm{C})^2$$
$$s + t + u = m_\mathrm{A}^2 + m_\mathrm{B}^2 + m_\mathrm{C}^2 + m_\mathrm{D}^2$$



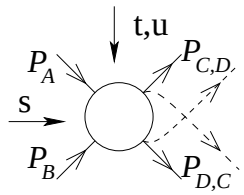


$$s = 2m(m+E)$$



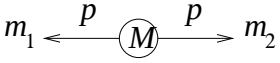
$$s = 4(p^2 + m^2)$$
$$t = -2p^2(1 - \cos\theta)$$
$$u = -2p^2(1 + \cos\theta)$$

Crossing Symmetry



	s-channel	t-channel	u-channel
	$A+B\to C+D$	$A+\bar{C}\to\bar{B}+D$	$A+\bar{D}\to\bar{B}+C$
s	$(P_\mathrm{A}+P_\mathrm{B})^2$	$(P_\mathrm{A}-P_\mathrm{C})^2$	$(P_\mathrm{A}-P_\mathrm{D})^2$
t	$(P_\mathrm{A}-P_\mathrm{C})^2$	$(P_\mathrm{A}+P_\mathrm{B})^2$	$(P_\mathrm{A}+P_\mathrm{B})^2$
u	$(P_\mathrm{A}-P_\mathrm{D})^2$	$(P_\mathrm{A}-P_\mathrm{D})^2$	$(P_\mathrm{A}-P_\mathrm{C})^2$

Two Body Decay



$$E_1=\frac{M^2+m_1^2-m_2^2}{2M}$$

$$E_2=\frac{M^2+m_2^2-m_1^2}{2M}$$

$$T_1=\Delta M\left(1-\frac{m_1}{M}-\frac{\Delta M}{2M}\right)\qquad \Delta M=M-m_1-m_2$$

$$P=\frac{\lambda^{1/2}(M^2,m_1^2,m_2^2)}{2M}\qquad \lambda(a,b,c)=a^2+b^2+c^2-2ab-2bc-2ca$$

$$\lambda(\alpha^2,\beta^2,\gamma^2)=\{\alpha^2-(\beta+\gamma)^2\}\{\alpha^2-(\beta-\gamma)^2\}$$

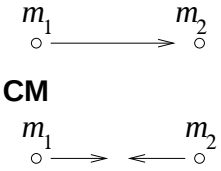
Two Body Reaction

$$s=m_1^2+m_2^2+2m_2E_1$$

$$E_1^*=\frac{s+m_1^2-m_2^2}{2\sqrt{s}}$$

$$E_2^*=\frac{s-m_1^2+m_2^2}{2\sqrt{s}}=\frac{m_2(m_2+E_1)}{\sqrt{s}}$$

$$P^*=\frac{\lambda^{1/2}(s,m_1^2,m_2^2)}{2\sqrt{s}}=\frac{m_2P_1}{\sqrt{s}}$$



In Lab system

$p_1^\mu=(E_1;P_1)$

$p_2^\mu=(m_2;0)$

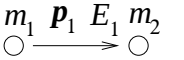
Three Body Decay $\quad M(\mathrm{rest})\to m_1+m_2+m_3$

$$m_{12}^2\equiv(P_1+P_2)^2=M^2-2ME_3+m_3^2\qquad E_3=\frac{M^2-m_{12}^2+m_3^2}{2M}$$

$$P_3=\frac{\lambda^{1/2}(M^2,m_{12}^2,m_3^2)}{2M}\qquad \cos\theta_{12}=\frac{P_3^2-P_1^2-P_2^2}{2P_1P_2}$$

Transformation between Lab and CM-Systems

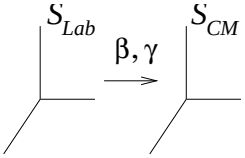
$$\gamma=\frac{E_1+m_2}{\sqrt{s}}\qquad \beta=\frac{P_1}{E_1+m_2}$$

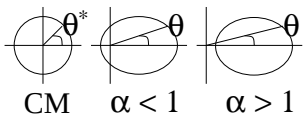


Energy: E_1+m_2 Momentum: P_1 Mass: $\sqrt{s}$

$$\begin{pmatrix}E^*\\P_\parallel^*\end{pmatrix}=\begin{pmatrix}\gamma & -\gamma\beta\\ -\gamma\beta & \gamma\end{pmatrix}\begin{pmatrix}E\\P_\parallel\end{pmatrix}\qquad \begin{pmatrix}E\\P_\parallel\end{pmatrix}=\begin{pmatrix}\gamma & \gamma\beta\\ \gamma\beta & \gamma\end{pmatrix}\begin{pmatrix}E^*\\P_\parallel^*\end{pmatrix}$$

*:CM-system





$$\tan\theta=\frac{\sin\theta^*}{\gamma(\alpha+\cos\theta^*)}\qquad \alpha\equiv\frac{\beta}{v^*_{\mathrm{particle}}/c}$$

$$\cos\theta^* = \frac{\pm\sqrt{D} - \alpha\gamma^2\tan^2\theta}{1+\gamma^2\tan^2\theta} \qquad \text{if } \alpha < 1, \text{ positive sign only}$$

$$D \equiv 1 + \gamma^2(1-\alpha^2)\tan^2\theta$$

Lorentz Invariant Phase Space

$$d^4P\delta(M^2-P^2)\theta(P^0) = \frac{d^3P}{2E} = \frac{P}{2}dEd\Omega \qquad E = \sqrt{\vec{P}^2+M^2}$$

$$d^3P = P^2dPd\Omega = PEdEd\Omega$$

Phase Spce of n-bodies (normalization $\int_{\text{unit}V}\rho\, dV = 2E$)

Phase space volume of single particle: $h^3 = (2\pi\hbar)^3 = (2\pi)^3$

$$d\Phi_n(P;P_1,\cdots,P_n) = (2\pi)^4\delta^4\left(P - \sum_{i=1}^{n} P_i\right)\prod_{i=1}^{n}\frac{d^3P_i}{(2\pi)^3 2E_i}$$

$M(\text{rest}) \to m_1 + m_2$

$$d\Phi_2(M;P_1,P_2) = \frac{1}{16\pi^2}\frac{|\vec{P}_1|}{M}d\Omega_1$$

$M(\text{rest}) \to m_1 + m_2 + m_3$ (In case no special direction in the initial state)

$$d\Phi_3(M;P_1,P_2,P_3) = \frac{1}{4(2\pi)^3}dE_1dE_2 = \frac{1}{16(2\pi)^3M^2}dm_{12}^2dm_{23}^2$$

$$\boxed{d\Gamma = \frac{|\mathfrak{M}|^2}{2M}d\Phi_n(P;P_1,\cdots,P_n)}$$

2-bodies decay $\qquad d\Gamma = \frac{1}{32\pi^2}|\mathfrak{M}|^2\frac{|\vec{P}_1|}{M^2}d\Omega_1$

3-bodies decay $\qquad d\Gamma = \frac{1}{(2\pi)^3}\frac{1}{8M}|\mathfrak{M}|^2dE_1dE_2$

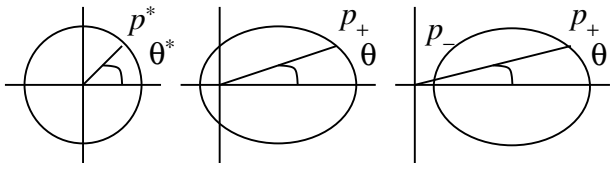
$$\boxed{d\sigma = \frac{|\mathfrak{M}|^2}{4\sqrt{(P_1\cdot P_2)^2 - m_1^2m_2^2}}d\Phi_n(P_1+P_2;P_3,\cdots,P_{n+2})}$$

m_1 $\boldsymbol{p}_1$ ○ ⟶ ○ m_2 $\qquad \sqrt{(P_1\cdot P_2)^2 - m_1^2m_2^2} = m_2P_1$

m_1 $\boldsymbol{p}_1$ ○ ⟶ ⟵ ○ m_2 $-\boldsymbol{p}_1$ $\qquad \sqrt{(P_1\cdot P_2)^2 - m_1^2m_2^2} = P_1\sqrt{s}$

$$d\Phi_2(P_1+P_2;P_3,P_4) = \frac{1}{16\pi^2}\frac{|\vec{P}_3|}{\sqrt{s}}d\Omega_3 \qquad \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{P}_3|}{|\vec{P}_1|}|\mathfrak{M}|^2$$

Jacobian of $\frac{d\sigma}{d\Omega_{\text{CM}}} \Leftrightarrow \frac{d\sigma}{d\Omega_{\text{Lab}}}$



CM $\qquad \alpha < 1 \qquad \alpha > 1$

$$\alpha \equiv \frac{v^*_{\text{Target}}}{v^*_{\text{Scattered}}}$$

$$\frac{d\Omega}{d\Omega^*_\pm} = \pm\sqrt{D}\cos\theta\left(\frac{P^*}{P_\pm}\right)^2 \qquad D \equiv 1+\gamma^2(1-\alpha^2)\tan^2\theta$$

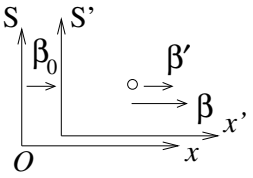
$$\left(\frac{P^*}{P_\pm}\right)^2 = \frac{\cos^2\theta}{\gamma^2}\left(\frac{1+\gamma^2\tan^2\theta}{\alpha\pm\sqrt{D}}\right)^2$$

Rapidity

$$y \equiv \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right) \qquad \beta = \tanh y \qquad \gamma = \cosh y$$

$$y = y_0 + y' \qquad \frac{dN}{dy} = \frac{dN}{dy'} \qquad \beta = \frac{\beta_0 + \beta'}{1 + \beta_0\beta'}$$

$$y \equiv \frac{1}{2}\ln\left(\frac{E+p_z}{E-p_z}\right)$$



$$\eta = -\ln\left(\tan\frac{\theta}{2}\right)\text{: pseudo rapidity} \qquad \cos\theta = \tanh\eta$$