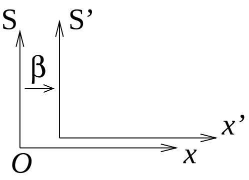
## Lorentz Transformation

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \cosh\varphi & -\sinh\varphi \\ -\sinh\varphi & \cosh\varphi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\varphi \qquad \gamma^2\beta^2 = \gamma^2 - 1 = \sinh^2\varphi$$

$$\gamma\beta = \sinh\varphi \ (\varphi > 0) \qquad \beta = \tanh\varphi \qquad \varphi\text{: Rapidity}$$




## $\beta/\gamma$ of Particle

$$\vec{\beta} = \frac{\vec{p}}{E} \qquad \gamma = \frac{E}{m} \qquad \vec{\beta}\gamma = \frac{\vec{p}}{m}$$

$$\alpha \equiv \frac{\beta\gamma}{\gamma+1} = \tanh\left(\frac{\varphi}{2}\right) \qquad \alpha^2 = \frac{\gamma-1}{\gamma+1} \qquad 1-\alpha^2 = \frac{2}{\gamma+1} \qquad 1+\alpha^2 = \frac{2\gamma}{\gamma+1}$$

$$\gamma = \frac{1+\alpha^2}{1-\alpha^2} \qquad \gamma+1 = \frac{2}{1-\alpha^2} \qquad \beta = \frac{2\alpha}{1+\alpha^2} \qquad \gamma(1+\beta) = \frac{1+\alpha}{1-\alpha}$$

General Lorentz transformation

$$a^\mu = (a^0; \vec{a}) \text{ in } S \qquad a'^\mu = (a'^0; \vec{a}') \text{ in } S' \qquad p^\mu = (p^0; \vec{p}) \text{ : 4-mom. for } S \to S'$$

$$a'^0 = (p^0 a^0 - \vec{p}\cdot\vec{a})/m = (a^0 - \vec{\beta}\cdot\vec{a})\gamma$$

$$\vec{a}' = \vec{a} + \frac{1}{m}\left(\frac{\vec{p}\cdot\vec{a}}{m+p^0} - a^0\right)\vec{p} = \vec{a} + \left(\frac{\gamma}{1+\gamma}\vec{\beta}\cdot\vec{a} - a^0\right)\vec{\beta}\gamma \qquad \text{where } m = \sqrt{p^2}$$

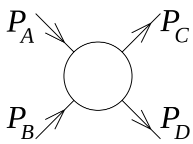
Boost

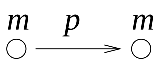
$$a^0 = (p^0 a'^0 + \vec{p}\cdot\vec{a}')/m = (a'^0 + \vec{\beta}\cdot\vec{a}')\gamma$$

$$\vec{a} = \vec{a}' + \frac{1}{m}\left(\frac{\vec{p}\cdot\vec{a}'}{m+p^0} + a'^0\right)\vec{p} = \vec{a}' + \left(\frac{\gamma}{1+\gamma}\vec{\beta}\cdot\vec{a}' + a'^0\right)\vec{\beta}\gamma$$

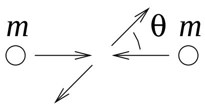
## Invariant Variables (Mandelstam variables)

$$s = (P_\mathrm{A} + P_\mathrm{B})^2 = (P_\mathrm{C} + P_\mathrm{D})^2$$
$$t = (P_\mathrm{A} - P_\mathrm{C})^2 = (P_\mathrm{B} - P_\mathrm{D})^2$$
$$u = (P_\mathrm{A} - P_\mathrm{D})^2 = (P_\mathrm{B} - P_\mathrm{C})^2$$
$$s + t + u = m_\mathrm{A}^2 + m_\mathrm{B}^2 + m_\mathrm{C}^2 + m_\mathrm{D}^2$$



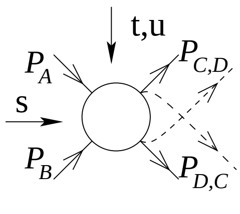




$$s = 2m(m+E)$$




$$s = 4(p^2 + m^2)$$
$$t = -2p^2(1-\cos\theta)$$
$$u = -2p^2(1+\cos\theta)$$

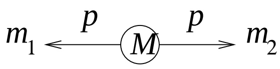
## Crossing Symmetry




| | $s$-channel | $t$-channel | $u$-channel |
|---|---|---|---|
| | $A+B \to C+D$ | $A+\bar{C} \to \bar{B}+D$ | $A+\bar{D} \to \bar{B}+C$ |
| $s$ | $(P_\mathrm{A}+P_\mathrm{B})^2$ | $(P_\mathrm{A}-P_\mathrm{C})^2$ | $(P_\mathrm{A}-P_\mathrm{D})^2$ |
| $t$ | $(P_\mathrm{A}-P_\mathrm{C})^2$ | $(P_\mathrm{A}+P_\mathrm{B})^2$ | $(P_\mathrm{A}+P_\mathrm{B})^2$ |
| $u$ | $(P_\mathrm{A}-P_\mathrm{D})^2$ | $(P_\mathrm{A}-P_\mathrm{D})^2$ | $(P_\mathrm{A}-P_\mathrm{C})^2$ |

## Two Body Decay

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}$$




$$E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M}$$

$$T_1 = \Delta M\left(1 - \frac{m_1}{M} - \frac{\Delta M}{2M}\right) \qquad \Delta M = M - m_1 - m_2$$

$$P = \frac{\lambda^{1/2}(M^2, m_1^2, m_2^2)}{2M} \qquad \lambda(a,b,c) = a^2 + b^2 + c^2 - 2a\,b - 2b\,c - 2c\,a$$

$$\lambda(\alpha^2, \beta^2, \gamma^2) = \{\alpha^2 - (\beta+\gamma)^2\}\{\alpha^2 - (\beta-\gamma)^2\}$$

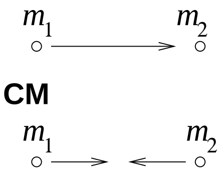
**Two Body Reaction**

$$s = m_1^2 + m_2^2 + 2m_2E_1$$

$$E_1^* = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}$$

$$E_2^* = \frac{s - m_1^2 + m_2^2}{2\sqrt{s}} = \frac{m_2(m_2 + E_1)}{\sqrt{s}}$$

$$P^* = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}} = \frac{m_2 P_1}{\sqrt{s}}$$




In Lab system

$p_1^\mu = (E_1; P_1)$

$p_2^\mu = (m_2; 0)$

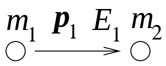
**Three Body Decay** $\quad M(\mathrm{rest}) \to m_1 + m_2 + m_3$

$$m_{12}^2 \equiv (P_1 + P_2)^2 = M^2 - 2ME_3 + m_3^2 \qquad E_3 = \frac{M^2 - m_{12}^2 + m_3^2}{2M}$$

$$P_3 = \frac{\lambda^{1/2}(M^2, m_{12}^2, m_3^2)}{2M} \qquad \cos\theta_{12} = \frac{P_3^2 - P_1^2 - P_2^2}{2P_1P_2}$$

**Transformation between Lab and CM-Systems**

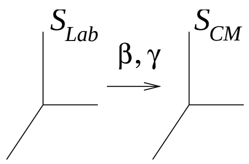
$$\gamma = \frac{E_1 + m_2}{\sqrt{s}} \qquad \beta = \frac{P_1}{E_1 + m_2}$$

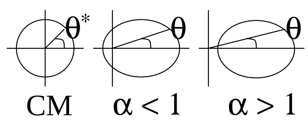



Energy: $E_1 + m_2$ $\qquad$ Momentum: $P_1$ $\qquad$ Mass: $\sqrt{s}$

$$\begin{pmatrix} E^* \\ P_\parallel^* \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} E \\ P_\parallel \end{pmatrix} \qquad \begin{pmatrix} E \\ P_\parallel \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} E^* \\ P_\parallel^* \end{pmatrix}$$

*:CM-system




$$\tan\theta = \frac{\sin\theta^*}{\gamma(\alpha + \cos\theta^*)} \qquad \alpha \equiv \frac{\beta}{v^*_{\mathrm{particle}}/c}$$

$$\cos\theta^* = \frac{\pm\sqrt{D} - \alpha\gamma^2\tan^2\theta}{1 + \gamma^2\tan^2\theta} \qquad \text{if } \alpha < 1, \text{ positive sign only}$$

$$D \equiv 1 + \gamma^2(1 - \alpha^2)\tan^2\theta$$

**Lorentz Invariant Phase Space**

$$d^4P\delta(M^2 - P^2)\theta(P^0) = \frac{d^3P}{2E} = \frac{P}{2}dEd\Omega \qquad E = \sqrt{\vec{P}^2 + M^2}$$

$$d^3P = P^2dPd\Omega = PEdEd\Omega$$

Phase Spce of n-bodies (normalization $\int_{\mathrm{unit}\,V}\rho\, dV = 2E$)

Phase space volume of single particle: $h^3 = (2\pi\hbar)^3 = (2\pi)^3$

$$d\Phi_n(P; P_1, \cdots, P_n) = (2\pi)^4\delta^4\left(P - \sum_{i=1}^{n} P_i\right)\prod_{i=1}^{n}\frac{d^3P_i}{(2\pi)^3 2E_i}$$

$M(\mathrm{rest}) \to m_1 + m_2$

$$d\Phi_2(M; P_1, P_2) = \frac{1}{16\pi^2}\frac{|\vec{P}_1|}{M}\, d\Omega_1$$

$M(\mathrm{rest}) \to m_1 + m_2 + m_3$ (In case no special direction in the initial state)

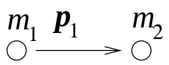
$$d\Phi_3(M; P_1, P_2, P_3) = \frac{1}{4(2\pi)^3}\, dE_1 dE_2 = \frac{1}{16(2\pi)^3 M^2}\, dm_{12}^2 dm_{23}^2$$

$$\boxed{d\Gamma = \frac{|\mathfrak{M}|^2}{2M}\, d\Phi_n(P; P_1, \cdots, P_n)}$$

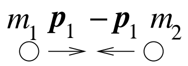
2-bodies decay $\quad d\Gamma = \frac{1}{32\pi^2}|\mathfrak{M}|^2\frac{|\vec{P}_1|}{M^2}d\Omega_1$

3-bodies decay $\quad d\Gamma = \frac{1}{(2\pi)^3}\frac{1}{8M}|\mathfrak{M}|^2 dE_1 dE_2$

$$\boxed{d\sigma = \frac{|\mathfrak{M}|^2}{4\sqrt{(P_1\cdot P_2)^2 - m_1^2 m_2^2}}\, d\Phi_n(P_1 + P_2; P_3, \cdots, P_{n+2})}$$



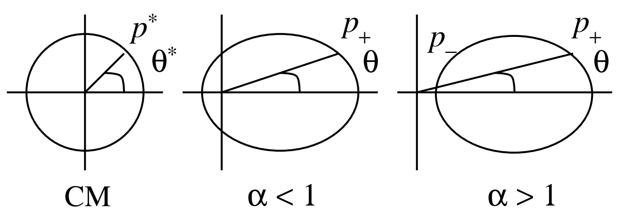

$$\sqrt{(P_1\cdot P_2)^2 - m_1^2 m_2^2} = m_2 P_1$$




$$\sqrt{(P_1\cdot P_2)^2 - m_1^2 m_2^2} = P_1\sqrt{s}$$

$$d\Phi_2(P_1 + P_2; P_3, P_4) = \frac{1}{16\pi^2}\frac{|\vec{P}_3|}{\sqrt{s}}d\Omega_3 \qquad \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{P}_3|}{|\vec{P}_1|}|\mathfrak{M}|^2$$

**Jacobian of $\frac{d\sigma}{d\Omega}_{\mathrm{CM}} \Leftrightarrow \frac{d\sigma}{d\Omega}_{\mathrm{Lab}}$**




$$\alpha \equiv \frac{v^*_{\mathrm{Target}}}{v^*_{\mathrm{Scattered}}}$$

$$\frac{d\Omega}{d\Omega^*_\pm} = \pm\sqrt{D}\cos\theta\left(\frac{P^*}{P_\pm}\right)^2 \qquad D \equiv 1 + \gamma^2(1-\alpha^2)\tan^2\theta$$

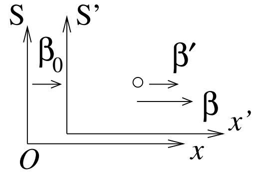
$$\left(\frac{P^*}{P_\pm}\right)^2 = \frac{\cos^2\theta}{\gamma^2}\left(\frac{1+\gamma^2\tan^2\theta}{\alpha \pm \sqrt{D}}\right)^2$$

**Rapidity**

$$y \equiv \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right) \qquad \beta = \tanh y \qquad \gamma = \cosh y$$

$$y = y_0 + y' \qquad \frac{dN}{dy} = \frac{dN}{dy'} \qquad \beta = \frac{\beta_0 + \beta'}{1 + \beta_0\beta'}$$

$$y \equiv \frac{1}{2}\ln\left(\frac{E + p_z}{E - p_z}\right)$$




$\eta = -\ln\left(\tan\frac{\theta}{2}\right)$: pseudo rapidity $\qquad \cos\theta = \tanh\eta$