

**Method of Lagrange Multipliers**

Theorem

$$D \subset \mathbb{R}^n \quad f: D \rightarrow \mathbb{R}^q \quad (C^r\text{-class}) \quad r \geq 1$$

$$h: D \rightarrow \mathbb{R} \quad (C^r\text{-class})$$

Consider that  $a \in D$  gives an extremum of  $h(x)$  on the condition  $f(x) = 0$  ( $x \in D$ ).If rank of  $(df)_a$  is  $q$ , then

$$\exists \lambda: \mathbb{R}^q \rightarrow \mathbb{R} : \text{linear} \quad (dh)_a = \lambda \circ (df)_a$$

We let  $f = (f_1, \dots, f_q)$ 

$$\exists \lambda_1, \dots, \lambda_q \quad \partial_i h(a) = \sum_{j=1}^q \lambda_j \partial_i f_j(a)$$

 $\lambda_1, \dots, \lambda_q$  : Undetermined multipliers**Residue**

$$f(z) \rightarrow \sum_{n=-\infty}^{\infty} a_n (z-a)^n : \text{Laurent expansion} \quad a_{-1} : \text{Residue at } a$$

$$D \subset C : \gamma = \partial D \quad \text{Single closing curve}$$

$$f(z) : \text{Regular on } \bar{D} \text{ except } a_1, \dots, a_n$$

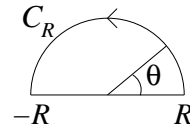
$$\frac{1}{2\pi i} \int_{\gamma} f(z) dz = \sum_{i=1}^n \text{Res}(f, a_i) \quad \text{Integrated forward direction of } D$$

- i. Pole of 1st order at  $a \in D$   $\text{Res}(f, a) = \lim_{z \rightarrow a} (z-a)f(z)$
- ii. Pole of n-th order at  $a \in D$   $g(z) \equiv (z-a)^n f(z)$   $\text{Res}(f, a) = \frac{g^{(n-1)}(a)}{(n-1)!}$

Lemma

$$\left| \int_{C_R} \frac{e^{iz}}{z} dz \right| \leq \int_0^{\pi} \frac{e^{-R \sin \theta}}{R} R d\theta = 2 \int_0^{\pi/2} e^{-R \sin \theta} d\theta$$

$$< 2 \int_0^{\pi/2} e^{-(2R/\pi)\theta} d\theta = \frac{\pi}{R} (1 - e^{-R}) \xrightarrow{R \rightarrow \infty} 0$$

**Curvature**

$$\mathbf{x} = \mathbf{x}(t) \quad t: \text{Parameter}$$

$$\text{We rewrite } \mathbf{x} \text{ with } s(t) = \int \left| \frac{d\mathbf{x}}{dt} \right| dt. \quad \frac{ds}{dt} = \left| \frac{d\mathbf{x}}{dt} \right| \quad \mathbf{x} = \mathbf{x}(s)$$

$$\mathbf{t}(s) = \frac{d\mathbf{x}}{ds} : \text{Unit tangential vector}$$

$$\frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{d\theta} \cdot \frac{d\theta}{ds} = \kappa \mathbf{n}$$

$\mathbf{n}$  : principal normal vector  
 $\kappa$  : curvature  
 $\rho \equiv \kappa^{-1}$  : radius of curvature

**Variational Method**

$$I = \int_D g(x_i, \varphi, \partial_i \varphi) dx^n \quad \varphi = \varphi(x_i) \quad D \subset \mathbb{R}^n$$

$$\varphi \rightarrow \varphi + \delta \varphi \quad \delta \varphi = 0 \text{ on } \partial D$$

$$\delta I = 0 \quad \Rightarrow \quad \frac{\partial g}{\partial \varphi} - \partial_i \frac{\partial g}{\partial \partial_i \varphi} = 0$$

Undetermined Multipliers

$$\int_D h(x_i, \varphi, \partial_i \varphi) = l \quad : \text{Constraint}$$

$$f \equiv g + \lambda h \quad \frac{\partial f}{\partial \varphi} - \partial_i \frac{\partial f}{\partial \partial_i \varphi} = 0$$

Solution :  $\varphi_\lambda = \varphi(x_i, \lambda)$

$$\int_D h(x_i, \varphi_\lambda, \partial_i \varphi_\lambda) = l \quad \Rightarrow \quad \exists \lambda$$

### Vector Operation

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\mathbf{A} = e(\mathbf{A} \cdot e) + e \times (\mathbf{A} \times e) \quad e : \text{unit vector}$$

$$\nabla(\varphi\psi) = \varphi \nabla \psi + \psi \nabla \varphi$$

$$\nabla \cdot (\varphi \mathbf{A}) = \mathbf{A} \cdot \nabla \varphi + \varphi \nabla \cdot \mathbf{A}$$

$$\nabla \times (\varphi \mathbf{A}) = \nabla \varphi \times \mathbf{A} + \varphi \nabla \times \mathbf{A}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla^2(\varphi\psi) = \varphi \nabla^2 \psi + \psi \nabla^2 \varphi + 2(\nabla \varphi \cdot \nabla \psi)$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\nabla \times (\nabla \varphi) = 0$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\mathbf{A} \times (\nabla \times \mathbf{A}) = -(\mathbf{A} \cdot \nabla) \mathbf{A} + \frac{1}{2} \nabla(\mathbf{A}^2)$$

$$\nabla f(\varphi) = \frac{df}{d\varphi} \nabla \varphi$$

$$\nabla^2 f(\varphi) = \frac{df}{d\varphi} \nabla^2 \varphi + \frac{d^2 f}{d\varphi^2} (\nabla \varphi)^2$$

$\varepsilon_{ijk} \varepsilon_{ljk} = 2\delta_{il}$ $\varepsilon_{ijk} \varepsilon_{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$ $\varepsilon_{ijk} = \begin{cases} 1 & (i,j,k)=(1,2,3) \\ & \text{even} \\ -1 & \text{odd} \\ 0 & \text{otherwise} \end{cases}$
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Position Vector  $\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\nabla \times \mathbf{r} = 0$$

$$\nabla \cdot \mathbf{r} = 3$$

$$\nabla |\mathbf{r}| = \hat{\mathbf{r}}$$

$\mathbf{A}$  : Constant vector

$$\nabla(\mathbf{A} \cdot \mathbf{r}) = \mathbf{A}$$

$$\nabla \times (\mathbf{A} \times \mathbf{r}) = 2\mathbf{A}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{r}) = 0$$

### Theorems of Vector Integral

Gauss' theorem  $\int_V \nabla \cdot \mathbf{A} dv = \int_{\partial V} \mathbf{A} \cdot d\mathbf{s}$

Green's theorem  $\int_V (\varphi \nabla^2 \psi - \psi \nabla^2 \varphi) dv = \int_{\partial V} \left( \varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n} \right) ds$

When  $\varphi = \frac{1}{r}$ ,  $\nabla^2 \psi = -\rho$ , this is a solution of Poisson eq.

Stokes's theorem  $\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\mathbf{t}$

### Jacobian

$$\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \dots & \frac{\partial f_1}{\partial x_n} \\ \dots & \dots & & & \dots \\ \frac{\partial f_n}{\partial x_1} & \dots & \dots & \dots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

## Curvilinear Coordinates

### Cylindrical Coordinates

$$\begin{aligned} \nabla \phi &= \frac{\partial \phi}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{\partial \phi}{\partial z} \hat{\mathbf{z}} & \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{pmatrix} &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \end{pmatrix} \\ \nabla \cdot \mathbf{A} &= \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \\ \nabla \times \mathbf{A} &= \left( \frac{1}{r} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) \hat{\mathbf{r}} + \left( \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \hat{\boldsymbol{\theta}} + \left( \frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \right) \hat{\mathbf{z}} \\ \nabla^2 \phi &= \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \end{aligned}$$

### Spherical Coordinates

$$\begin{aligned} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} &= \begin{pmatrix} \sin \theta \cos \varphi & \cos \theta \cos \varphi & -\sin \varphi \\ \sin \theta \sin \varphi & \cos \theta \sin \varphi & \cos \varphi \\ \cos \theta & -\sin \theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} \\ \nabla \phi &= \frac{\partial \phi}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \hat{\boldsymbol{\varphi}} \\ \nabla \cdot \mathbf{A} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\varphi}{\partial \varphi} \\ \nabla \times \mathbf{A} &= \frac{1}{r \sin \theta} \left\{ \frac{\partial}{\partial \theta} (\sin \theta A_\varphi) - \frac{\partial A_\theta}{\partial \varphi} \right\} \hat{\mathbf{r}} + \frac{1}{r} \left\{ \frac{1}{\sin \theta} \frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r} (r A_\varphi) \right\} \hat{\boldsymbol{\theta}} \\ &\quad + \frac{1}{r} \left\{ \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right\} \hat{\boldsymbol{\varphi}} \\ \nabla^2 \phi &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2} \end{aligned}$$

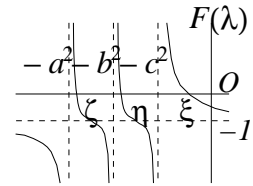
### Ellipsoidal Coordinates

$$F(\lambda) = \frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} - 1 = 0 \quad [a > b > c]$$

: family of confocal quadric surfaces

Among the family, there are 3 surfaces including a given point  $(x, y, z)$  and let their  $\lambda$ 's be  $\xi, \eta, \zeta$   $[-a^2 < \zeta < -b^2 < \eta < -c^2 < \xi]$ .

$\lambda = \xi, \eta$ , and  $\zeta$  give an ellipsoid, hyperboloid of one sheet and of two sheets, respectively. This set of  $(\xi, \eta, \zeta)$  gives a point in the ellipsoidal coordinates, which corresponds to  $(x, y, z)$ .



$$\begin{aligned} x &= \pm \sqrt{\frac{(\xi + a^2)(\eta + a^2)(\zeta + a^2)}{(b^2 - a^2)(c^2 - a^2)}} & y &= \pm \sqrt{\frac{(\xi + b^2)(\eta + b^2)(\zeta + b^2)}{(c^2 - b^2)(a^2 - b^2)}} \\ z &= \pm \sqrt{\frac{(\xi + c^2)(\eta + c^2)(\zeta + c^2)}{(a^2 - c^2)(b^2 - c^2)}} \end{aligned}$$

$$\begin{aligned} \nabla^2 \phi &= \frac{4}{(\xi - \eta)(\xi - \zeta)(\eta - \zeta)} \left[ (\eta - \zeta) R_\xi \frac{\partial}{\partial \xi} \left( R_\xi \frac{\partial \phi}{\partial \xi} \right) + (\zeta - \xi) R_\eta \frac{\partial}{\partial \eta} \left( R_\eta \frac{\partial \phi}{\partial \eta} \right) \right. \\ &\quad \left. + (\xi - \eta) R_\zeta \frac{\partial}{\partial \zeta} \left( R_\zeta \frac{\partial \phi}{\partial \zeta} \right) \right] \end{aligned}$$

$$R_s \equiv \sqrt{(s+a^2)(s+b^2)(s+c^2)} \quad s = \xi, \eta, \zeta$$

#### Curvilinear Coordinates

$$u_k = f_k(x_1, \dots, x_n) \quad g_{ik} \equiv \sum_{r=1}^n \frac{\partial x_r}{\partial u_i} \frac{\partial x_r}{\partial u_k}$$

#### Orthogonal Curvilinear Coordinates

$$\text{Orthogonal curvilinear coordinates} \implies g_{ik} \equiv \sum_{r=1}^n \frac{\partial x_r}{\partial u_i} \frac{\partial x_r}{\partial u_k} = 0 \quad (i \neq k)$$

#### Measure coefficient

$$h_k \equiv \sqrt{\sum_{r=1}^n \left( \frac{\partial x_r}{\partial u_k} \right)^2} \quad g_k \equiv \sqrt{\sum_{r=1}^n \left( \frac{\partial x_r}{\partial u_k} \right)^2} \quad h_k g_k = 1$$

$$\text{grad } \phi = \sum_{k=1}^n h_k \frac{\partial \phi}{\partial u_k} \mathbf{e}_k \quad \Delta \phi = h_1 \dots h_n \left[ \sum_{k=1}^n \frac{\partial}{\partial u_k} \left( \frac{h_k^2}{h_1 \dots h_n} \frac{\partial \phi}{\partial u_k} \right) \right]$$

$\mathbf{X} = \sum X_k \mathbf{e}_k$  : Vector field

$$\text{div } \mathbf{X} = h_1 \dots h_n \left[ \sum_{k=1}^n \frac{\partial}{\partial u_k} \left( \frac{h_k X_k}{h_1 \dots h_n} \right) \right]$$

#### Orthogonal Curvilinear Coordinates in 2 Dimension

$$u = u(x, y) \quad v = v(x, y)$$

$$\text{Orthogonal condition} \quad \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} = 0$$

$$\text{Conjugate coordinates} \quad u + i v = f(x + i y)$$

$$\text{Cauchy-Riemann} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$h_1 = h_2 = \sqrt{\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2} = \sqrt{\left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2}$$

#### Examples

$f(z)$	Transformation	$h_1 = h_2$	Coordinates
$\log z$	$x = e^u \cos v$ $y = e^u \sin v$	$\frac{1}{\sqrt{x^2 + y^2}}$	Polar coordinates
$z^2$	$u = x^2 - y^2$ $v = 2xy$	$2\sqrt{x^2 + y^2}$	Rectangular hyperbolic coordinates
$\sqrt{2z}$	$x = \frac{1}{2}(u^2 - v^2)$ $y = uv$	$\frac{1}{\sqrt{u^2 + v^2}}$	Parabolic coordinates
$2i \coth^{-1} \frac{z}{a}$	$x = \frac{a \sinh v}{\cosh v - \cos u}$ $y = \frac{a \sin u}{\cosh v - \cos u}$	$\frac{\cosh v - \cos u}{a}$	Hyperbolic coordinates
$\cosh^{-1} \frac{z}{c}$	$x = c \cosh u \cos v$ $y = c \sinh u \sin v$	$\frac{1}{\sqrt{\sinh^2 u + \sin^2 v}}$	Elliptic coordinates

#### General Rotational Body Coordinates

Curvilinear coordinates given by rotating orthogonal coordinates  $z = f(u, v)$ ,  $r = g(u, v)$  in  $(z, r)$  plane around  $z$ -axis. ( $r^2 = x^2 + y^2$ )

$$x = g(u, v) \cos \phi \quad y = g(u, v) \sin \phi \quad z = f(u, v)$$

$$h_u = \frac{1}{\sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial u}\right)^2}} \quad h_v = \frac{1}{\sqrt{\left(\frac{\partial f}{\partial v}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2}} \quad h_\varphi = \frac{1}{r}$$

### Trigonometric Functions

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \quad \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \quad \cot(\alpha + \beta) = \frac{\cot \alpha \cot \beta - 1}{\cot \alpha + \cot \beta}$$

$$a \sin \alpha + b \cos \alpha = \sqrt{a^2 + b^2} \sin(\alpha + \gamma) \quad \text{where } \tan \gamma = b/a$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha \quad \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha$$

$$\sin^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2} \quad \cos^2 \frac{\alpha}{2} = \frac{1 + \cos \alpha}{2} \quad \tan^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{1 + \cos \alpha}$$

$$\sin \alpha \cos \beta = \frac{1}{2} \{ \sin(\alpha + \beta) + \sin(\alpha - \beta) \}$$

$$\sin \alpha \sin \beta = -\frac{1}{2} \{ \cos(\alpha + \beta) - \cos(\alpha - \beta) \} \quad \cos \alpha \cos \beta = \frac{1}{2} \{ \cos(\alpha + \beta) + \cos(\alpha - \beta) \}$$

$$\sin \alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \quad \cos \alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

### Hyperbolic Function

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh x = \frac{e^x + e^{-x}}{2} \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cosh x = \frac{1}{\sqrt{1 - \tanh^2 x}}$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\sinh^{-1} x = \log(x + \sqrt{x^2 + 1}) \quad \cosh^{-1} x = \pm \log(x + \sqrt{x^2 - 1})$$

$$\sinh^2 \frac{x}{2} = \frac{\cosh x - 1}{2} \quad \cosh^2 \frac{x}{2} = \frac{\cosh x + 1}{2}$$

$$\tanh \frac{x}{2} = \frac{\tanh x}{1 + \sqrt{1 - \tanh^2 x}}$$

### Elliptic Integral

When  $f(x)$  is a polynomial of 3rd or 4th order, the following integrals result in normal forms

$$\int \frac{dx}{\sqrt{f(x)}} \quad \int \frac{x^2}{\sqrt{f(x)}} dx \quad \int \frac{dx}{(1 + ax^2)\sqrt{f(x)}}$$

Normal forms

$$\text{Type I} \quad F(\varphi, k) = \int_0^\varphi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^{\sin \varphi} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}$$

$$\text{Type II} \quad E(\varphi, k) = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \theta} d\theta = \int_0^{\sin \varphi} \sqrt{\frac{1 - k^2 z^2}{1 - z^2}} dz$$

$$\begin{aligned}\text{Type III} \quad \Pi(\varphi; c, k) &= \int_0^\varphi \frac{d\theta}{(1 + c \sin^2 \theta) \sqrt{1 - k^2 \sin^2 \theta}} \\ &= \int_0^{\sin \varphi} \frac{dz}{(1 + c z^2) \sqrt{(1 - z^2)(1 - k^2 z^2)}}\end{aligned}$$

Complete Elliptic Integral

$$\begin{aligned}K(k) &= F\left(\frac{\pi}{2}, k\right) = \frac{\pi}{2} \left\{ \sum_{n=0}^{\infty} \left[ \frac{(2n-1)!!}{(2n)!!} \right]^2 k^{2n} \right\} \\ E(k) &= E\left(\frac{\pi}{2}, k\right) = \frac{\pi}{2} \left\{ 1 - \sum_{n=1}^{\infty} \left[ \frac{(2n-1)!!}{(2n)!!} \right]^2 \frac{k^{2n}}{2n-1} \right\}\end{aligned}$$

**Legendre Polynomial**

$$(1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0 \quad \text{Legendre differential equation}$$

$$\begin{aligned}P_n(x) &= \frac{(2n-1)!!}{n!} \left[ x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} x^{n-4} - \dots \right] \\ &= \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n\end{aligned}$$

$$P_0(x) = 1 \quad P_1(x) = x = \cos \theta$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) = \frac{1}{4}(3 \cos 2\theta - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) = \frac{1}{8}(5 \cos 3\theta + 3 \cos \theta)$$

Legendre functions of the second kind

$$\begin{aligned}Q_n(x) &= \frac{n!}{(2n+1)!!} \left[ \frac{1}{x^{n+1}} + \frac{(n+1)(n+2)}{2(2n+3)} \frac{1}{x^{n+3}} + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4 \cdot (2n+3)(2n+5)} \frac{1}{x^{n+5}} + \dots \right] \\ &= \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[ (x^2 - 1)^n \log \frac{1+x}{1-x} \right] - \frac{1}{2} P_n(x) \log \frac{1+x}{1-x} \quad (|x| > 1)\end{aligned}$$

$$\frac{1}{\sqrt{1-2hx+h^2}} = \begin{cases} \sum_{n=0}^{\infty} h^n P_n(x) & (|h| < \min |x \pm \sqrt{x^2-1}|) \\ \sum_{n=0}^{\infty} \frac{1}{h^{n+1}} P_n(x) & (|h| > \max |x \pm \sqrt{x^2-1}|) \end{cases}$$

$$\int_{-1}^1 x^m P_n(x) dx = \begin{cases} 0 & (m < n) \\ \frac{2 \cdot n!}{(2n+1)!!} & (m = n) \end{cases} \quad \int_{-1}^1 P_n(x) P_m(x) dx = \begin{cases} \frac{2}{2n+1} & (m = n) \\ 0 & (m \neq n) \end{cases}$$

**Associated Legendre Function**

$$(1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \left\{ n(n+1) - \frac{m^2}{1-x^2} \right\} y = 0$$

$$P_n^m(x) = (1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m} \quad Q_n^m(x) = (1-x^2)^{m/2} \frac{d^m Q_n(x)}{dx^m}$$

$$\int_{-1}^1 P_n^m(x) P_l^m(x) dx = \begin{cases} 0 & (n \neq l) \\ \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} & (n = l) \end{cases}$$

$$\int_{-1}^1 \frac{P_n^m(x) P_l^l(x)}{1-x^2} dx = \begin{cases} 0 & (m \neq l) \\ \frac{1}{m} \frac{(n+m)!}{(n-m)!} & (m = l) \end{cases}$$

$$P_1^1(x) = (1 - x^2)^{1/2} = \sin \theta$$

$$P_2^1(x) = 3(1 - x^2)^{1/2}x = \frac{3}{2}\sin 2\theta$$

$$P_2^2(x) = 3(1 - x^2) = \frac{3}{2}(1 - \cos 2\theta)$$

$$P_3^1(x) = \frac{3}{2}(1 - x^2)^{1/2}(5x^2 - 1) = \frac{1}{8}(\sin \theta + 5\sin 3\theta)$$

$$P_3^2(x) = 15(1 - x^2)x = \frac{15}{4}(\cos \theta - \cos 3\theta)$$

$$P_3^3(x) = 15(1 - x^2)^{3/2} = \frac{15}{4}(3\sin \theta - \sin 3\theta)$$

### Bessel Function

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)y = 0$$

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\nu + n + 1)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$$N_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$$

$$\Gamma(\xi) \equiv \int_0^\infty e^{-t} t^{\xi-1} dt$$

$$J_{-n}(x) = (-1)^n J_n(x)$$

### Spherical Bessel Function

$$\frac{d^2y}{dx^2} + \frac{2}{x} \frac{dy}{dx} + \left[1 - \frac{l(l+1)}{x^2}\right]y = 0$$

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+1/2}(x) = (-x)^l \left(\frac{1}{x} \frac{d}{dx}\right)^l \left(\frac{\sin x}{x}\right)$$

$$n_l(x) = \sqrt{\frac{\pi}{2x}} N_{l+1/2}(x) = -(-x)^l \left(\frac{1}{x} \frac{d}{dx}\right)^l \left(\frac{\cos x}{x}\right)$$

### Asymptotic Formulas $x \gg 1$

$$J_\nu \cong \sqrt{\frac{2}{\pi x}} \cos \left\{x - \frac{\pi}{2} \left(\nu + \frac{1}{2}\right)\right\}$$

$$j_l(x) \cong \frac{1}{x} \sin \left(x - \frac{l\pi}{2}\right)$$

$$N_\nu \cong \sqrt{\frac{2}{\pi x}} \sin \left\{x - \frac{\pi}{2} \left(\nu + \frac{1}{2}\right)\right\}$$

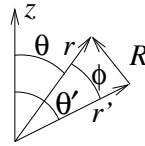
$$n_l(x) \cong -\frac{1}{x} \cos \left(x - \frac{l\pi}{2}\right)$$

### Expansion Formulas

$$e^{ikr \cos \theta} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos \theta)$$

$$\frac{e^{ikR}}{R} = i k \sum_{l=0}^{\infty} (2l+1) j_l(kr') h_l^{(1)}(kr) P_l(\cos \phi)$$

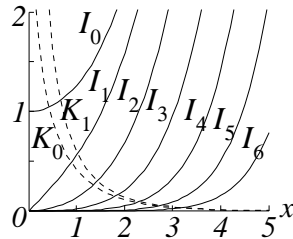
$$h_l^{(1)}(x) \equiv j_l(x) + i n_l(x)$$



### Modified Bessel Function

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)y = 0$$

$$I_\nu(x), K_\nu(x) : \text{Solutions}$$



### Hermite Differential Equation

$$x \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + n y = 0$$

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} \quad \text{One of two solutions}$$

$$\text{Generating function :} \quad \exp \left\{ \frac{x^2}{2} - \frac{(x-y)^2}{2} \right\} = \sum_{n=0}^{\infty} \frac{y^n}{n!} H_n(x)$$

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-\frac{x^2}{2}} dx = \begin{cases} n! \sqrt{2\pi} & (m=n) \\ 0 & (m \neq n) \end{cases}$$

### Laguerre Differential Equation

$$x \frac{d^2 y}{dx^2} + (1-x) \frac{dy}{dx} + n y = 0$$

$$L_n(x) = \frac{e^x}{n!} \left( \frac{d}{dx} \right)^n (e^{-x} x^n) = \sum_{r=0}^n (-1)^r \binom{n}{r} \frac{x^r}{r!}$$

$$\text{Orthogonality :} \quad \int_0^{\infty} e^{-x} L_m(x) L_n(x) dx = \begin{cases} 1 & (m=n) \\ 0 & (m \neq n) \end{cases}$$

Associated Laguerre polynomial

$$x \frac{d^2 y}{dx^2} + (k+1-x) \frac{dy}{dx} + n y = 0$$

$$\begin{aligned} L_n^k(x) &= \frac{x^{-k} e^x}{n!} \left( \frac{d}{dx} \right)^n (e^{-x} x^{n+k}) = \sum_{r=0}^n (-1)^r \binom{n+k}{n-r} \frac{x^r}{r!} \\ &= \sum_{r=0}^n (-1)^r \frac{(n+k)!}{(n-r)!(k+r)!} \frac{x^r}{r!} \end{aligned}$$

$$L_0^k(x) = 1 \quad L_1^k(x) = -x + k + 1 \quad L_2^k(x) = \frac{x^2}{2} - (k+2)x + \frac{(k+2)(k+1)}{2}$$

$$L_3^k(x) = -\frac{x^3}{6} + \frac{(k+3)x^2}{2} - \frac{(k+2)(k+3)x}{2} + \frac{(k+1)(k+2)(k+3)}{6}$$

### Integral

$$\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx$$

$f(x)$	$F(x) = \int f(x) dx$
$\frac{1}{x^2 + a^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1} \frac{x}{a}$
$\sqrt{a^2 - x^2}$	$\frac{1}{2} \left( x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right)$
$\sqrt{x^2 \pm a^2}$	$\frac{1}{2} \left( x \sqrt{x^2 \pm a^2} \pm a^2 \log  x + \sqrt{x^2 \pm a^2}  \right)$
$\frac{1}{\sqrt{x^2 + a^2}}$	$\log  x + \sqrt{x^2 + a^2}  = \sinh^{-1} \frac{x}{a}$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\log  x + \sqrt{x^2 - a^2}  = \cosh^{-1} \frac{x}{a}$

### Rational Function

$$f(x) = \frac{Q(x)}{P(x)} \longrightarrow f(x) = R(x) + \sum_{i=1}^r \sum_{k=1}^{m_i} \frac{A_{ik}}{(x-a_i)^k} + \sum_{j=1}^s \sum_{l=1}^{n_j} \frac{B_{jl}x + C_{jl}}{(x^2 + \beta_j x + \gamma_j)^l}$$



: Partial fraction expansion

$$\int \frac{dx}{(x-\alpha)^n} = \begin{cases} \log|x-\alpha| & (n=1) \\ -\frac{1}{n-1} \frac{1}{(x-\alpha)^{n-1}} & (n>1) \end{cases}$$

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^m} dx = \frac{B}{2} \int \frac{2x+\beta}{(x^2+\beta x+\gamma)^m} dx + C' \int \frac{dx}{(x^2+\beta x+\gamma)^m}$$

$$\text{where } C' = C - B \frac{\beta}{2}$$

$$\int \frac{2x+\beta}{(x^2+\beta x+\gamma)^m} dx = \begin{cases} \log(x^2+\beta x+\gamma) & (m=1) \\ -\frac{1}{m-1} \frac{1}{(x^2+\beta x+\gamma)^{m-1}} & (m>1) \end{cases}$$

$$\int \frac{dx}{(x^2+\beta x+\gamma)^m} = \int \frac{dt}{(t^2+\beta'^2)^m} \quad t = x + \frac{\beta}{2} \quad \beta'^2 = \gamma - \frac{\beta^2}{4}$$

We define  $I_n \equiv \int \frac{dx}{(x^2+a^2)^n}$

$$I_n = \frac{1}{2a^2(n-1)} \frac{x}{(x^2+a^2)^{n-1}} + \frac{2n-3}{a^2(2n-2)} I_{n-1}$$

**Rational Function of Trigonometric Functions**

We let  $P(X, Y)$  be a rational function of  $X, Y$ , and consider

$$\int P(\cos x, \sin x) dx$$

We define  $t = \tan \frac{x}{2}$

$$\cos x = \frac{1-t^2}{1+t^2} \quad \sin x = \frac{2t}{1+t^2} \quad \frac{dt}{dx} = \frac{1+t^2}{2}$$

$$\int P(\cos x, \sin x) dx = \int P\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \frac{2}{1+t^2} dt$$

which results in an integral of a rational function of  $t$ .

$$\text{Sup.)} \quad I_n = \int \sin^n x dx \quad S_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$$

$$I_n = \frac{1}{n} (-\cos x \sin^{n-1} x + (n-1) I_{n-2}) \quad I_0 = x \quad I_1 = -\cos x$$

$$S_n = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & (n \geq 2: \text{even}) \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} & (n \geq 3: \text{odd}) \end{cases} \quad S_0 = \frac{\pi}{2} \quad S_1 = 1$$

**Rational Function of Quadratic Irrational Functions**

$$\int P(x, \sqrt{ax^2+bx+c}) dx$$

i. Case of  $a > 0$

$$\sqrt{ax^2+bx+c} = t - \sqrt{a}x \quad x = \frac{t^2 - c}{2\sqrt{a}t + b}$$

The integrand goes to a rational function of  $t$ .

ii. Case of  $a < 0$

$$ax^2 + bx + c = a(x - \alpha)(x - \beta) \quad \alpha, \beta \in \mathbb{R}$$

$$t = \sqrt{\frac{a(x - \alpha)}{x - \beta}}$$

The integrand goes to a rational function of  $t$ .

Change of Variables

$$\int \cdots \int_{\Omega} f(x_1 \cdots x_n) dx_1 \cdots dx_n = \int \cdots \int_{\Delta} f(\varphi_1(\xi_1 \cdots \xi_n), \dots, \varphi_n(\xi_1 \cdots \xi_n)) \left| \frac{\partial(\varphi_1 \cdots \varphi_n)}{\partial(\xi_1 \cdots \xi_n)} \right| d\xi_1 \cdots d\xi_n$$

Other Integrals

$$\int_0^\infty e^{-ax^2} x^{2n} dx = \frac{(2n-1)!!}{2^{n+1}} \sqrt{\frac{\pi}{a^{2n+1}}} \quad \int_0^\infty e^{-ax^2} dx = \frac{\sqrt{\pi}}{2\sqrt{a}}$$

$$\int_0^\infty e^{-ax^2} x^{2n+1} dx = \frac{n!}{2a^{n+1}}$$

**First-Order Differential Equation**

Variables-separable  $y' = g(y)f(x)$

Equation	Change of Variables	New Equation
$y' = f(ax + by + c)$	$u = ax + by + c$	$\frac{du}{dx} = a + bf(u)$
$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ (Homogeneous)	$u = \frac{y}{x}$	$\frac{du}{dx} = \frac{f(u) - u}{x}$
$y' = f\left(\frac{ax + by + c}{a'x + b'y + c'}\right)$ ( $a'b' - a'b \neq 0$ )	$\begin{cases} a\alpha + b\beta + c = 0 \\ a'\alpha + b'\beta + c' = 0 \end{cases}$ $x = u + \alpha, y = v + \beta$	$\frac{dv}{du} = f\left(\frac{a + b\frac{v}{u}}{a' + b'\frac{v}{u}}\right)$
( $a'b' - a'b = 0$ )	$\exists \alpha: a = \alpha a', b = \alpha b'$	$y' = f\left(\alpha + \frac{c - \alpha c'}{a'x + b'y + c'}\right)$

**First-Order Linear Differential Equation**

$$y' + a(x)y = b(x)$$

$$p(x) = - \int_{x_0}^x a(x) dx \quad u(x) = \int_{x_0}^x b(x) e^{-p(x)} dx$$

$$\implies y(x) = u(x)e^{p(x)} + Ce^{p(x)}$$

Sup.) An equation including derived functions of unknown  $y$  but not independent of variable  $x$ .

$$F(y, y', \dots, y^{(n)}) = 0$$

If we let  $y' = p$

$$y'' = \frac{dp}{dy} p, \quad y''' = \frac{d^2 p}{dy^2} p^2 + \left(\frac{dp}{dy}\right)^2 p, \quad \dots$$

which result in differential equations of unknown function  $p$  and independent variable  $y$ .

**Linear Ordinary Differential Equation with Constant Coefficients**

Homogeneous Equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2y'' + a_1y' + a_0y = f(x)$$

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_2D^2 + a_1D + a_0)y = f(x) \quad \text{where } D \equiv \frac{d}{dx}$$

$$P(D)y = f(x)$$

Homogeneous equation

$$P(D)y = 0$$

$$\alpha_1 \cdots \alpha_n \in C : \quad \text{Solution of } P(t) = 0$$

$$(D - \alpha_1)(D - \alpha_2) \cdots (D - \alpha_n)y = 0$$

Fundamental solutions

$$(D - \alpha_i)y = 0 \quad \implies \quad y = C_i e^{\alpha_i x}$$

$$(D - \alpha_j)^m y = 0 \quad \implies \quad y = (C_0^j + C_1^j x + \cdots + C_{m-1}^j x^{m-1}) e^{\alpha_j x}$$

$$(\text{Let } y = e^{\alpha x} u)$$

### Method of Variation of Parameters

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

If fundamental solutions of its homogeneous equation  $y_1 y_2 \cdots y_n$  are known,

$$\text{Wronskian} \quad W(y_1, \cdots, y_n) \equiv \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$\text{Green function} \quad G(x, s) \equiv \frac{1}{W(y_1(s), \cdots, y_n(s))} \begin{vmatrix} y_1(s) & \cdots & y_n(s) \\ \vdots & \ddots & \vdots \\ y_1^{(n-2)}(s) & \cdots & y_n^{(n-2)}(s) \\ y_1(x) & \cdots & y_n(x) \end{vmatrix}$$

The solution of the non-homogeneous equation is

$$y(x) = \sum_{j=0}^{n-1} c_j y_j(x) + \int_{x_0}^x G(x, s) f(s) ds$$

### Lagrange's Differential Equation

$$p(x, y, u) \frac{\partial u}{\partial x} + q(x, y, u) \frac{\partial u}{\partial y} = r(x, y, u)$$

$$\text{Auxiliary equation} \quad \frac{dx}{p(x, y, u)} = \frac{dy}{q(x, y, u)} = \frac{du}{r(x, y, u)}$$

We let two independent solutions of this equation be

$$g(x, y, u) = a \quad h(x, y, u) = b$$

General solutions of Lagrange's equation are any functions of  $g$  and  $h$ .

$$f(g, h) = 0$$

### Laplace Equation in 2 Dimension

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\xi = x + i y \quad \eta = x - i y \quad \phi(x, y) = \tilde{\phi}(\xi, \eta)$$

$$\nabla^2 \phi = 4 \frac{\partial^2}{\partial \xi \partial \eta} \tilde{\phi} = 0$$

$$\text{General solutions} \quad \tilde{\phi}(\xi, \eta) = f_1(\xi) + f_2(\eta)$$

$$\text{Rotating symmetry (Function of } \xi \eta = x^2 + y^2) \quad g(\xi \eta) = a \ln \xi \eta + b$$

## Fourier Transform

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk$$

### General Formulas in Fourier Transform

$$\int_{-\infty}^{\infty} \hat{f}(k) \hat{g}^*(k) dk = \int_{-\infty}^{\infty} f(x) g^*(x) dx \quad (\text{Parseval's identity})$$

$$f(x) \quad \longleftrightarrow \quad \hat{f}(k)$$

$$f^*(x) \quad \longleftrightarrow \quad \hat{f}^*(-k)$$

$$f\left(\frac{x}{a} + b\right) \quad \longleftrightarrow \quad |a| e^{iabk} \hat{f}(ak)$$

$$f(ax) e^{-ibx} \quad \longleftrightarrow \quad \frac{1}{|a|} \hat{f}\left(\frac{k+b}{a}\right)$$

$$x^n f(x) \quad \longleftrightarrow \quad (i)^n \frac{d^n \hat{f}(k)}{dk^n}$$

$$f^{(n)}(x) \quad \longleftrightarrow \quad (ik)^n \hat{f}(k)$$

$$(f * g)(x) \quad \longleftrightarrow \quad \hat{f}(k) \hat{g}(k)$$

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t) g(t) dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t) dt$$

### Examples of Fourier Transform

$$f(x) \quad \longleftrightarrow \quad \hat{f}(k)$$

$$\frac{1}{x^2 + a^2} \quad (a > 0) \quad \longleftrightarrow \quad \sqrt{\frac{\pi}{2}} \frac{e^{-a|k|}}{a}$$

$$\frac{1}{\sqrt{x^2 + a^2}} \quad \longleftrightarrow \quad \sqrt{\frac{2}{\pi}} K_0(a|k|) \quad (\text{Modified Bessel})$$

$$e^{-ax^2} \quad \longleftrightarrow \quad \frac{1}{\sqrt{2a}} e^{-\frac{k^2}{4a}}$$

$$\tan^{-1} \frac{x}{a} \quad \longleftrightarrow \quad -i \sqrt{\frac{\pi}{2}} \frac{e^{-a|k|}}{k}$$

$$\delta(x) \quad \longleftrightarrow \quad \frac{1}{\sqrt{2\pi}}$$

$$\frac{1}{x} \quad \longleftrightarrow \quad -i \sqrt{\frac{\pi}{2}} \text{sgn}(k)$$

$$\theta(x) \quad \text{Heviside step function} \quad \longleftrightarrow \quad \sqrt{\frac{\pi}{2}} \delta(k) - \frac{i}{\sqrt{2\pi} k} = \frac{1}{i\sqrt{2\pi}} \frac{1}{k - i\epsilon}$$

$$e^{-ax} \theta(x) \quad \longleftrightarrow \quad \frac{1}{\sqrt{2\pi} (a + ik)}$$

$$\square(x) \equiv \theta\left(x + \frac{1}{2}\right) - \theta\left(x - \frac{1}{2}\right) \quad \longleftrightarrow \quad \sqrt{\frac{2}{\pi}} \frac{\sin\left(\frac{k}{2}\right)}{k}$$

### Delta Functions

$$\delta(x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$$

$$\delta(ax) = \frac{\delta(x)}{|a|} \quad x\delta(x) = 0 \quad x\delta'(x) = -\delta(x)$$

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|} \quad f(x_i) = 0$$

$$\lim_{\epsilon \rightarrow +0} \frac{1}{x \pm i\epsilon} = \text{V.P.} \frac{1}{x} \mp i\pi\delta(x)$$

## Integral Equation

Abel's Problem

$$f(x) = \int_{-\infty}^{\infty} K(x-y)u(y)dy \quad u(x): \text{Unknown function}$$

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{f}(k)}{\hat{K}(k)} e^{ikx} dk$$

$K(x-y)$  : Kernel

$$f(x) = g(x) + \lambda \int_{-\infty}^{\infty} K(x-y)f(y)dy \quad f(x): \text{unknown function}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{g}(k)}{1 - \sqrt{2\pi}\lambda\hat{K}(k)} e^{ikx} dk$$

## Green Functions

Consider a partial differential equation of type  $Lu(x) = f(x)$ .

$$L = F(D_{x_1}, \dots, D_{x_n}) \equiv F(D_{x_i})$$

$$F(D_{x_i})G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \quad \mathbf{r} = (x_1, \dots, x_n)$$

Then

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{(2\pi)^n} \int \frac{e^{ik(\mathbf{r}-\mathbf{r}')}}{F(i\mathbf{k})} d\mathbf{k}$$

Diffusion Equation in 1 Dimension

$$L_{xy} = D_x^2 - D_y$$

$$G(\mathbf{r} - \mathbf{r}') = \begin{cases} -\frac{1}{\sqrt{4\pi|y-y'|}} \exp\left\{-\frac{(x-x')^2}{4|y-y'|}\right\} & (y > y') \\ 0 & (y < y') \end{cases}$$

Initial-value problem

$$(D_x^2 - D_t)u(x, t) = 0 \quad \text{with } u(x, 0) = f(x)$$

$$(D_x^2 - D_t)u(x, t) = -\delta(t)f(x)$$

Wave Equation in 1 Dimension

$$L_{x,y} = D_x^2 - D_y^2$$

$$G(\mathbf{r} - \mathbf{r}') = \begin{cases} \frac{1}{2}\theta(y - |x|) & (y > 0) \\ 0 & (y < 0) \end{cases}$$

Laplace's Equation in 2 Dimension

$$L_{x,y} = D_x^2 + D_y^2$$

$$G(\mathbf{r} - \mathbf{r}') = \frac{1}{2\pi} \ln \left( \frac{1}{R} \right) \quad R \equiv |\mathbf{r} - \mathbf{r}'|$$

Sup.) Poisson integral formula (Dirichlet problem)

Cauchy integral formula

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R_0^2 - r^2)u(R_0, \phi)}{R_0^2 - 2R_0r \cos(\theta - \phi) + r^2} d\phi$$

$$f(z) = \frac{1}{2\pi i} \int_c \frac{f(\zeta)}{\eta - z} d\zeta$$

Laplace's Equation in 3 Dimension

$$L_{x,y,z} = D_x^2 + D_y^2 + D_z^2$$

$$G(\mathbf{r} - \mathbf{r}') = -\frac{1}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

Diffusion Equation in 3 Dimension

$$L_{x,y,z,t} = D_x^2 + D_y^2 + D_z^2 - D_t$$

$$G(\mathbf{r} - \mathbf{r}', t - t') = \frac{1}{[4\pi(t - t')]^{3/2}} \exp \left[ -\frac{(\mathbf{r} - \mathbf{r}')^2}{4|t - t'|} \right]$$

Wave Equation in 3 Dimension

$$L_{x,y,z,t} = D_x^2 + D_y^2 + D_z^2 - D_t^2,$$

$$G(\mathbf{r} - \mathbf{r}', t - t') = \begin{cases} \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \delta(t - t' - |\mathbf{r} - \mathbf{r}'|) & (t - t' > 0) \\ 0 & (t - t' < 0) \end{cases}$$

Sup.) Consider  $L_{\mathbf{r},t} = F(\Delta, D_t)$ . We let Green function of  $L_{x,t} = F(D_x^2, D_t)$  be  $g_1(x, y)$ .

Then Green function of  $L_{\mathbf{r},t}$  is

$$G(\mathbf{r}, t) = -\frac{1}{2\pi r} D_r g_1(r, t) \quad r = |\mathbf{r}|$$

where  $g_1$  is even function about  $r$ , therefore  $D_r g_1$  is odd.

**Laplace Transform**

$$\tilde{f}(s) \equiv \int_0^\infty f(x) e^{-sx} dx$$

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \tilde{f}(s) ds$$

$c > \sigma$  : Region of convergence

$$f(x) \longleftrightarrow$$

$$\tilde{f}(s)$$

$$f(\alpha x)$$

$$\frac{1}{\alpha} \tilde{f}\left(\frac{s}{\alpha}\right) \quad (\alpha > 0)$$

$$\begin{cases} f(x-a) & (x \geq a) \\ 0 & (x < a) \end{cases}$$

$$e^{-sa} \tilde{f}(s)$$

$$f(x) e^{bx}$$

$$\tilde{f}(s-b) \quad (s > \sigma + b)$$

$$x^n f(x)$$

$$\left(-\frac{d}{ds}\right)^n \tilde{f}(s)$$

$$\frac{1}{x^n} f(x)$$

$$\int_s^\infty \dots \int_s^\infty \tilde{f}(s) (ds)^n \quad (\text{nth order integral})$$

$$f'(x)$$

$$-f(0) + s \tilde{f}(s)$$

$$f^{(n)}(x) \qquad s^n \tilde{f}(s) - \sum_{r=0}^{n-1} s^{n-r-1} f^{(r)}(0)$$

$$\int_0^x \cdots \int_0^x f(x) (dx)^n \qquad \frac{1}{s^n} \tilde{f}(s)$$

$$x^m f^{(n)}(x) \qquad (m \geq n) \qquad \left(-\frac{d}{ds}\right)^m [s^n \tilde{f}(s)]$$

$$\int_0^x f(x-y)g(y)dy \equiv (f * g)(x) \qquad \tilde{f}(s) \tilde{g}(s)$$

#### Examples of Laplace Transform

$$f(x) \qquad \longleftrightarrow \qquad \tilde{f}(s)$$

$$\delta(x) \qquad 1$$

$$\theta(x) \qquad \text{Heviside step function} \qquad \frac{1}{s}$$

$$x^\nu \qquad (-1 < \nu < \infty) \qquad \frac{\Gamma(\nu+1)}{s^{\nu+1}} \qquad n! = \Gamma(n+1)$$

$$e^{ax} \qquad \frac{1}{s-a}$$

$$e^{ax} x^{\nu-1} \qquad (\nu > 0) \qquad \frac{\Gamma(\nu)}{(s-a)^\nu}$$

$$\sin ax \qquad \frac{a}{s^2 + a^2}$$

$$\cos ax \qquad \frac{s}{s^2 + a^2}$$

$$J_0(2\sqrt{\lambda x}) \qquad \frac{1}{s} e^{-\lambda/s}$$