Method of Lagrange Multipliers

Theorem

$$D \subset \Re^n \qquad f\colon D \to \Re^q \qquad (C^r\text{-class}) \qquad r \geq 1$$

$$h\colon D \to \Re \qquad (C^r\text{-class})$$

Consider that $a \in D$ gives an extremum of $h(x)$ on the condition $f(x) = 0$ $(x \in D)$.

If rank of $(df)_a$ is q, then

$$^\exists\lambda\colon \Re^q \to \Re : \text{linear} \qquad (dh)_a = \lambda \circ (df)_a$$

We let $f = (f_1, \cdots, f_q)$

$$^\exists\lambda_1, \cdots, \lambda_q \qquad \partial_i h(a) = \sum_{j=1}^{q} \lambda_j \partial_i f_j(a)$$

$\lambda_1, \cdots, \lambda_q$: Undetermined multipliers

Residue

$$f(z) \to \sum_{n=-\infty}^{\infty} a_n (z-a)^n : \text{Laurent expansion} \qquad a_{-1} : \text{Residue at } a$$

$D \subset C$: $\gamma = \partial D$ Single closing curve

$f(z)$: Regular on $\bar{D}$ except $a_1, \cdots, a_n$

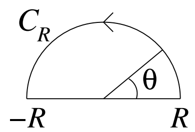
$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{i=1}^{n} \mathrm{Res}(f, a_i) \qquad \text{Integrated forward direction of } D$$

i. Pole of 1st order at $a \in D$ $\qquad \mathrm{Res}(f,a) = \lim_{z\to a}(z-a)f(z)$

ii. Pole of n-th order at $a \in D$ $\qquad g(z) \equiv (z-a)^n f(z) \qquad \mathrm{Res}(f,a) = \dfrac{g^{(n-1)}(a)}{(n-1)!}$

Lemma

$$\left|\int_{C_R} \frac{e^{iz}}{z} dz\right| \leq \int_0^\pi \frac{e^{-R\sin\theta}}{R} R\, d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta} d\theta$$
$$< 2\int_0^{\pi/2} e^{-(2R/\pi)\theta} d\theta = \frac{\pi}{R}(1 - e^{-R}) \underset{R\to\infty}{\to} 0$$



Curvature

$$\boldsymbol{x} = \boldsymbol{x}(t) \qquad t\text{: Parameter}$$

We rewrite $\boldsymbol{x}$ with $s(t) = \displaystyle\int \left|\frac{d\boldsymbol{x}}{dt}\right| dt$. $\qquad \dfrac{ds}{dt} = \left|\dfrac{d\boldsymbol{x}}{dt}\right| \qquad \boldsymbol{x} = \boldsymbol{x}(s)$

$$\boldsymbol{t}(s) = \frac{d\boldsymbol{x}}{ds} : \text{ Unit tangential vector}$$

$$\frac{d\boldsymbol{t}}{ds} = \frac{d\boldsymbol{t}}{d\theta} \cdot \frac{d\theta}{ds} = \kappa \boldsymbol{n}$$

$\boldsymbol{n}$: principal normal vector

κ : curvature

$\rho \equiv \kappa^{-1}$: radius of curvature

Variational Method

$$I = \int_D g(x_i, \varphi, \partial_i \varphi) dx^n \qquad \varphi = \varphi(x_i) \qquad D \subset \Re^n$$

$$\varphi \to \varphi + \delta\varphi \qquad \delta\phi = 0 \text{ on } \partial D$$

$$\delta I = 0 \qquad \Longrightarrow \qquad \frac{\partial g}{\partial \varphi} - \partial_i \frac{\partial g}{\partial \partial_i \varphi} = 0$$

Undetermined Multipliers

$$\int_D h(x_i, \varphi, \partial_i \varphi) = l \quad : \text{Constraint}$$

$$f \equiv g + \lambda h \qquad \frac{\partial f}{\partial \varphi} - \partial_i \frac{\partial f}{\partial \partial_i \varphi} = 0$$

$$\text{Solution : } \quad \varphi_\lambda = \varphi(x_i, \lambda)$$

$$\int_D h(x_i, \varphi_\lambda, \partial_i \varphi_\lambda) = l \quad \Longrightarrow \quad {}^\exists \lambda$$

Vector Operation

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$$

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$$

$$\boldsymbol{A} = \boldsymbol{e}(\boldsymbol{A} \cdot \boldsymbol{e}) + \boldsymbol{e} \times (\boldsymbol{A} \times \boldsymbol{e}) \qquad \boldsymbol{e} : \text{unit vector}$$

$$\boldsymbol{\nabla}(\varphi\psi) = \varphi \boldsymbol{\nabla} \psi + \psi \boldsymbol{\nabla} \varphi$$

$$\boldsymbol{\nabla} \cdot (\varphi \boldsymbol{A}) = \boldsymbol{A} \cdot \boldsymbol{\nabla} \varphi + \varphi \boldsymbol{\nabla} \cdot \boldsymbol{A}$$

$$\boldsymbol{\nabla} \times (\varphi \boldsymbol{A}) = \boldsymbol{\nabla} \varphi \times \boldsymbol{A} + \varphi \boldsymbol{\nabla} \times \boldsymbol{A}$$

$$\boldsymbol{\nabla}(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{A} + \boldsymbol{A} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) + \boldsymbol{B} \times (\boldsymbol{\nabla} \times \boldsymbol{A})$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B})$$

$$\boldsymbol{\nabla}^2(\varphi\psi) = \varphi \boldsymbol{\nabla}^2 \psi + \psi \boldsymbol{\nabla}^2 \varphi + 2(\boldsymbol{\nabla} \varphi \cdot \boldsymbol{\nabla} \psi)$$

$$\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A}$$

$$\begin{aligned}
&\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \varphi) = 0 \\
&\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times A) = 0 \\
&\boldsymbol{A} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = -(\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{A} + \frac{1}{2} \boldsymbol{\nabla}(\boldsymbol{A}^2) \\
&\boldsymbol{\nabla} f(\varphi) = \frac{df}{d\varphi} \boldsymbol{\nabla} \varphi \\
&\boldsymbol{\nabla}^2 f(\varphi) = \frac{df}{d\varphi} \boldsymbol{\nabla}^2 \varphi + \frac{d^2 f}{d\varphi^2} (\boldsymbol{\nabla} \varphi)^2
\end{aligned}$$

$$\boxed{\begin{aligned}
&\varepsilon_{ijk} \varepsilon_{ljk} = 2\delta_{il} \\
&\varepsilon_{ijk} \varepsilon_{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \\
&\varepsilon_{ijk} = \begin{cases} 1 & \overset{(i,j,k)=(1,2,3)}{\text{even}} \\ -1 & \text{odd} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}}$$

Position Vector $\quad \boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$

$$\boldsymbol{\nabla} \times \boldsymbol{r} = 0 \qquad \boldsymbol{\nabla} \cdot \boldsymbol{r} = 3 \qquad \boldsymbol{\nabla}|r| = \hat{\boldsymbol{r}}$$

$\boldsymbol{A}$: Constant vector

$$\boldsymbol{\nabla}(\boldsymbol{A} \cdot \boldsymbol{r}) = \boldsymbol{A} \qquad \boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{r}) = 2\boldsymbol{A} \qquad \boldsymbol{\nabla} \cdot (\boldsymbol{A} \times \boldsymbol{r}) = 0$$

Theorems of Vector Integral

Gauss' theorem $\quad \displaystyle\int_V \boldsymbol{\nabla} \cdot \boldsymbol{A} dv = \int_{\partial V} \boldsymbol{A} \cdot d\boldsymbol{s}$

Green's theorem $\quad \displaystyle\int_V (\varphi \boldsymbol{\nabla}^2 \psi - \psi \boldsymbol{\nabla}^2 \varphi) dv = \int_{\partial V} \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n} \right) ds$

When $\varphi = \dfrac{1}{r}$, $\boldsymbol{\nabla}^2 \psi = -\rho$, this is a solution of Poisson eq.

Stokes's theorem $\quad \displaystyle\int_S (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot d\boldsymbol{s} = \oint_C \boldsymbol{A} \cdot d\boldsymbol{t}$

Jacobian

$$\frac{\partial(f_1,\cdots,f_n)}{\partial(x_1,\cdots,x_n)}=\begin{vmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \cdots & \frac{\partial f_1}{\partial x_n}\\ \cdots & \cdots & & & \cdots\\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \cdots & \frac{\partial f_n}{\partial x_n}\end{vmatrix}$$

Curvilinear Coordinates

Cylindrical Coordinates

$$\nabla\phi=\frac{\partial\phi}{\partial r}\hat{\boldsymbol{r}}+\frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}}+\frac{\partial\phi}{\partial z}\hat{\boldsymbol{z}} \qquad \begin{pmatrix}\hat{\boldsymbol{x}}\\ \hat{\boldsymbol{y}}\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{r}}\\ \hat{\boldsymbol{\theta}}\end{pmatrix}$$

$$\nabla\cdot\boldsymbol{A}=\frac{1}{r}\frac{\partial}{\partial r}(r\,A_r)+\frac{1}{r}\frac{\partial A_\theta}{\partial\theta}+\frac{\partial A_z}{\partial z}$$

$$\nabla\times\boldsymbol{A}=\left(\frac{1}{r}\frac{\partial A_z}{\partial\theta}-\frac{\partial A_\theta}{\partial z}\right)\hat{\boldsymbol{r}}+\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\theta}}+\left(\frac{1}{r}\frac{\partial}{\partial r}(r\,A_\theta)-\frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{z}}$$

$$\nabla^2\phi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}$$

Spherical Coordinates

$$\begin{pmatrix}\hat{\boldsymbol{x}}\\ \hat{\boldsymbol{y}}\\ \hat{\boldsymbol{z}}\end{pmatrix}=\begin{pmatrix}\sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi\\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi\\ \cos\theta & -\sin\theta & 0\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol{r}}\\ \hat{\boldsymbol{\theta}}\\ \hat{\boldsymbol{\varphi}}\end{pmatrix}$$

$$\nabla\phi=\frac{\partial\phi}{\partial r}\hat{\boldsymbol{r}}+\frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}}+\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{\boldsymbol{\varphi}}$$

$$\nabla\cdot\boldsymbol{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta)+\frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

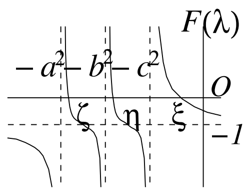
$$\begin{aligned}\nabla\times\boldsymbol{A}=\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta A_\varphi)-\frac{\partial A_\theta}{\partial\varphi}\right\}\hat{\boldsymbol{r}}&+\frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi}-\frac{\partial}{\partial r}(r\,A_\varphi)\right\}\hat{\boldsymbol{\theta}}\\ &+\frac{1}{r}\left\{\frac{\partial}{\partial r}(r\,A_\theta)-\frac{\partial A_r}{\partial\theta}\right\}\hat{\boldsymbol{\varphi}}\end{aligned}$$

$$\nabla^2\phi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

Ellipsoidal Coordinates

$$F(\lambda)=\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}-1=0 \qquad [a>b>c]$$

: family of confocal quadric surfaces



Among the family, there are 3 surfaces including a given point (x,y,z) and let their λ's be ξ,η,ζ $[-a^2<\zeta<-b^2<\eta<-c^2<\xi]$.
$\lambda=\xi$, η, and ζ give an ellipsoid, hyperboloid of one sheet and of two sheets, respectively. This set of (ξ,η,ζ) gives a point in the ellipsoidal coordinates, which corresponds to (x,y,z).

$$x=\pm\sqrt{\frac{(\xi+a^2)(\eta+a^2)(\zeta+a^2)}{(b^2-a^2)(c^2-a^2)}} \qquad y=\pm\sqrt{\frac{(\xi+b^2)(\eta+b^2)(\zeta+b^2)}{(c^2-b^2)(a^2-b^2)}}$$

$$z=\pm\sqrt{\frac{(\xi+c^2)(\eta+c^2)(\zeta+c^2)}{(a^2-c^2)(b^2-c^2)}}$$

$$\begin{aligned}\nabla^2\phi=\frac{4}{(\xi-\eta)(\xi-\zeta)(\eta-\zeta)}\bigg[(\eta-\zeta)R_\xi\frac{\partial}{\partial\xi}\left(R_\xi\frac{\partial\phi}{\partial\xi}\right)&+(\zeta-\xi)R_\eta\frac{\partial}{\partial\eta}\left(R_\eta\frac{\partial\phi}{\partial\eta}\right)\\ &+(\xi-\eta)R_\zeta\frac{\partial}{\partial\zeta}\left(R_\zeta\frac{\partial\phi}{\partial\zeta}\right)\bigg]\end{aligned}$$

$$R_s \equiv \sqrt{(s+a^2)(s+b^2)(s+c^2)} \qquad s=\xi,\eta,\zeta$$

Curvilinear Coordinates

$$u_k = f_k(x_1,\cdots,x_n) \qquad g_{ik} \equiv \sum_{r=1}^{n} \frac{\partial x_r}{\partial u_i}\frac{\partial x_r}{\partial u_k}$$

Orthogonal Curvilinear Coordinates

Orthogonal curvilinear coordinates $\Longrightarrow$ $g_{ik} \equiv \sum_{r=1}^{n} \frac{\partial x_r}{\partial u_i}\frac{\partial x_r}{\partial u_k} = 0 \quad (i \neq k)$

Measure coefficient

$$h_k \equiv \sqrt{\sum_{r=1}^{n}\left(\frac{\partial u_k}{\partial x_r}\right)^2} \qquad g_k \equiv \sqrt{\sum_{r=1}^{n}\left(\frac{\partial x_r}{\partial u_k}\right)^2} \qquad h_k g_k = 1$$

$$\mathrm{grad}\,\phi = \sum_{k=1}^{n} h_k \frac{\partial \phi}{\partial u_k}\boldsymbol{e}_k \qquad \triangle\phi = h_1\cdots h_n\left[\sum_{k=1}^{n}\frac{\partial}{\partial u_k}\left(\frac{h_k^2}{h_1\cdots h_n}\frac{\partial\phi}{\partial u_k}\right)\right]$$

$\boldsymbol{X} = \sum X_k\,\boldsymbol{e}_k$: Vector field

$$\mathrm{div}\boldsymbol{X} = h_1\cdots h_n\left[\sum_{k=1}^{n}\frac{\partial}{\partial u_k}\left(\frac{h_k X_k}{h_1\cdots h_n}\right)\right]$$

Orthogonal Curvilinear Coordinates in 2 Dimension

$$u = u(x,y) \qquad v = v(x,y)$$

Orthogonal condition $\quad \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0$

Conjugate coordinates $\quad u + i\,v = f(x + i\,y)$

Cauchy-Riemann $\quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$h_1 = h_2 = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2} = \sqrt{\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2}$$

Examples

$f(z)$	Transformation	$h_1 = h_2$	Coordinates
$\log z$	$x = e^u \cos v$ $y = e^u \sin v$	$\frac{1}{\sqrt{x^2+y^2}}$	Polar coordinates
z^2	$u = x^2 - y^2$ $v = 2x\,y$	$2\sqrt{x^2+y^2}$	Rectangular hyperbolic coordinates
$\sqrt{2z}$	$x = \frac{1}{2}(u^2 - v^2)$ $y = u\,v$	$\frac{1}{\sqrt{u^2+v^2}}$	Parabolic coordinates
$2i\coth^{-1}\frac{z}{a}$	$x = \frac{a\sinh v}{\cosh v - \cos u}$ $y = \frac{a\sin u}{\cosh v - \cos u}$	$\frac{\cosh v - \cos u}{a}$	Hyperbolic coordinates
$\cosh^{-1}\frac{z}{c}$	$x = c\cosh u\cos v$ $y = c\sinh u\sin v$	$\frac{1}{\sqrt{\sinh^2 u + \sin^2 v}}$	Elliptic coordinates

General Rotational Body Coordinates

Curvilinear coordinates given by rotating orthogonal coordinates $z = f(u,v)$, $r = g(u,v)$ in (z,r) plane around z-axis. $(r^2 = x^2 + y^2)$

$$x = g(u,v)\cos\phi \qquad y = g(u,v)\sin\phi \qquad z = f(u,v)$$

$$h_u = \frac{1}{\sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial u}\right)^2}} \qquad h_v = \frac{1}{\sqrt{\left(\frac{\partial f}{\partial v}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2}} \qquad h_\varphi = \frac{1}{r}$$

Trigonometric Functions

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} \qquad \cot(\alpha+\beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$$

$$a\sin\alpha + b\cos\alpha = \sqrt{a^2+b^2}\sin(\alpha+\gamma) \quad \text{where } \tan\gamma = b/a$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} \qquad \cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2} \qquad \tan^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{1+\cos\alpha}$$

$$\sin\alpha\cos\beta = \frac{1}{2}\{\sin(\alpha+\beta) + \sin(\alpha-\beta)\}$$

$$\sin\alpha\sin\beta = -\frac{1}{2}\{\cos(\alpha+\beta) - \cos(\alpha-\beta)\} \qquad \cos\alpha\cos\beta = \frac{1}{2}\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\}$$

$$\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \qquad \cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sec x = \frac{1}{\cos x} \qquad \operatorname{cosec} x = \csc x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x}$$

Hyperbolic Function

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad \cosh x = \frac{1}{\sqrt{1-\tanh^2 x}}$$

$$\sinh(x \pm y) = \sinh x\cosh y \pm \cosh x\sinh y$$

$$\cosh(x \pm y) = \cosh x\cosh y \pm \sinh x\sinh y$$

$$\sinh^{-1} x = \log(x + \sqrt{x^2+1}) \qquad \cosh^{-1} x = \pm\log(x + \sqrt{x^2-1})$$

$$\sinh^2\frac{x}{2} = \frac{\cosh x - 1}{2} \qquad \cosh^2\frac{x}{2} = \frac{\cosh x + 1}{2} \qquad \tanh\frac{x}{2} = \frac{\tanh x}{1 + \sqrt{1-\tanh^2 x}}$$

Elliptic Integral

When $f(x)$ is a polymonial of 3rd or 4th order, the following integrals result in normal forms

$$\int \frac{dx}{\sqrt{f(x)}} \qquad \int \frac{x^2}{\sqrt{f(x)}}\,dx \qquad \int \frac{dx}{(1+a\,x^2)\sqrt{f(x)}}$$

Normal forms

$$\text{Type I} \qquad F(\varphi,k) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin\varphi} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

$$\text{Type II} \qquad E(\varphi,k) = \int_0^{\varphi} \sqrt{1-k^2\sin^2\theta}\,d\theta = \int_0^{\sin\varphi} \sqrt{\frac{1-k^2z^2}{1-z^2}}\,dz$$

$$\text{Type III} \qquad \begin{aligned}\Pi(\varphi;c,k) &= \int_0^{\varphi} \frac{d\theta}{(1+c\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \\ &= \int_0^{\sin\varphi} \frac{dz}{(1+c\,z^2)\sqrt{(1-z^2)(1-k^2z^2)}}\end{aligned}$$

Complete Elliptic Integral

$$K(k) = F\Big(\frac{\pi}{2},k\Big) = \frac{\pi}{2}\left\{\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 k^{2n}\right\}$$

$$E(k) = E\Big(\frac{\pi}{2},k\Big) = \frac{\pi}{2}\left\{1-\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 \frac{k^{2n}}{2n-1}\right\}$$

Legendre Polynomial

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad \text{Legendre differential equation}$$

$$\begin{aligned}P_n(x) &= \frac{(2n-1)!!}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}x^{n-4} - \cdots\right] \\ &= \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\end{aligned}$$

$$P_0(x) = 1 \qquad P_1(x) = x = \cos\theta$$

$$P_2(x) = \frac{1}{2}(3x^2-1) = \frac{1}{4}(3\cos 2\theta - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3-3x) = \frac{1}{8}(5\cos 3\theta + 3\cos\theta)$$

Legendre functions of the second kind

$$\begin{aligned}Q_n(x) &= \frac{n!}{(2n+1)!!}\left[\frac{1}{x^{n+1}} + \frac{(n+1)(n+2)}{2(2n+3)}\frac{1}{x^{n+3}} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot(2n+3)(2n+5)}\frac{1}{x^{n+5}} + \cdots\right] \\ &= \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n \log\frac{1+x}{1-x}\right] - \frac{1}{2}P_n(x)\log\frac{1+x}{1-x} \qquad (|x|>1)\end{aligned}$$

$$\frac{1}{\sqrt{1-2h\,x+h^2}} = \begin{cases}\displaystyle\sum_{n=0}^{\infty} h^n P_n(x) & \Big(|h| < \min\Big|x \pm \sqrt{x^2-1}\Big|\Big) \\ \displaystyle\sum_{n=0}^{\infty} \frac{1}{h^{n+1}}P_n(x) & \Big(|h| > \max\Big|x \pm \sqrt{x^2-1}\Big|\Big)\end{cases}$$

$$\int_{-1}^{1} x^m P_n(x)dx = \begin{cases} 0 & (m<n) \\ \dfrac{2\cdot n!}{(2n+1)!!} & (m=n)\end{cases} \qquad \int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases}\dfrac{2}{2n+1} & (m=n) \\ 0 & (m\neq n)\end{cases}$$

Associated Legendre Function

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\}y = 0$$

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m P_n(x)}{dx^m} \qquad Q_n^m(x) = (1-x^2)^{m/2}\frac{d^m Q_n(x)}{dx^m}$$

$$\int_{-1}^{1} P_n^m(x)P_l^m(x)\,dx = \begin{cases} 0 & (n\neq l) \\ \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!} & (n=l)\end{cases}$$

$$\int_{-1}^{1} \frac{P_n^m(x)P_n^l(x)}{1-x^2}\,dx = \begin{cases} 0 & (m\neq l) \\ \dfrac{1}{m}\dfrac{(n+m)!}{(n-m)!} & (m=l)\end{cases}$$

$$P_1^1(x)=(1-x^2)^{1/2}=\sin\theta$$

$$P_2^1(x)=3(1-x^2)^{1/2}x=\frac{3}{2}\sin 2\theta \qquad \mathrm{P}_2^2(x)=3(1-x^2)=\frac{3}{2}(1-\cos 2\theta)$$

$$P_3^1(x)=\frac{3}{2}(1-x^2)^{1/2}(5x^2-1)=\frac{1}{8}(\sin\theta+5\sin 3\theta)$$

$$P_3^2(x)=15(1-x^2)x=\frac{15}{4}(\cos\theta-\cos 3\theta)$$

$$P_3^3(x)=15(1-x^2)^{3/2}=\frac{15}{4}(3\sin\theta-\sin 3\theta)$$

Bessel Function

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{\nu^2}{x^2}\right)y=0$$

$$J_\nu(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\nu+n+1)}\left(\frac{x}{2}\right)^{2n+\nu} \qquad N_\nu(x)=\frac{\cos\nu\pi J_\nu(x)-J_{-\nu}(x)}{\sin\nu\pi}$$

$$\Gamma(\xi)\equiv\int_0^\infty e^{-t}t^{\xi-1}dt$$

$$J_{-n}(x)=(-1)^nJ_n(x)$$

Spherical Bessel Function

$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+\left[1-\frac{l(l+1)}{x^2}\right]y=0$$

$$j_l(x)=\sqrt{\frac{\pi}{2x}}\mathrm{J}_{l+1/2}(x)=(-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\left(\frac{\sin x}{x}\right)$$

$$n_l(x)=\sqrt{\frac{\pi}{2x}}\mathrm{N}_{l+1/2}(x)=-(-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\left(\frac{\cos x}{x}\right)$$

Asymptotic Formulas $\qquad x\gg 1$

$$J_\nu\cong\sqrt{\frac{2}{\pi x}}\cos\left\{x-\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)\right\} \qquad j_l(x)\cong\frac{1}{x}\sin\left(x-\frac{l\pi}{2}\right)$$

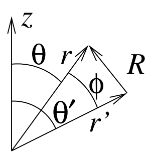
$$N_\nu\cong\sqrt{\frac{2}{\pi x}}\sin\left\{x-\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)\right\} \qquad n_l(x)\cong-\frac{1}{x}\cos\left(x-\frac{l\pi}{2}\right)$$

Expansion Formulas

$$e^{ikr\cos\theta}=\sum_{l=0}^{\infty}(2l+1)i^lj_l(kr)P_l(\cos\theta)$$

$$\frac{e^{ikR}}{R}=ik\sum_{l=0}^{\infty}(2l+1)j_l(kr')h_l^{(1)}(kr)P_l(\cos\phi)$$

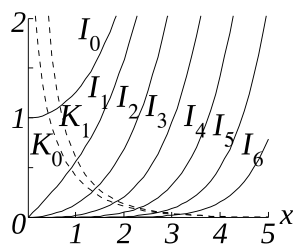
$$h_l^{(1)}(x)\equiv j_l(x)+i\,n_l(x)$$



Modified Bessel Function

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\left(1+\frac{\nu^2}{x^2}\right)y=0$$

$I_\nu(x),\ K_\nu(x)$: Solutions



Hermite Differential Equation

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n\,y = 0$$

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} \qquad \text{One of two solutions}$$

Generating function : $\exp\left\{\frac{x^2}{2} - \frac{(x-y)^2}{2}\right\} = \sum_{n=0}^{\infty} \frac{y^n}{n!} H_n(x)$

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-\frac{x^2}{2}} dx = \begin{cases} n!\sqrt{2\pi} & (m = n) \\ 0 & (m \neq n) \end{cases}$$

Laguerre Differential Equation

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + n\,y = 0$$

$$L_n(x) = \frac{e^x}{n!}\left(\frac{d}{dx}\right)^n (e^{-x} x^n) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{x^r}{r!}$$

Orthogonality : $\int_0^{\infty} e^{-x} L_m(x) L_n(x) dx = \begin{cases} 1 & (m = n) \\ 0 & (m \neq n) \end{cases}$

Associated Laguerre polynomial

$$x\frac{d^2y}{dx^2} + (k+1-x)\frac{dy}{dx} + n\,y = 0$$

$$\begin{aligned} L_n^k(x) &= \frac{x^{-k} e^x}{n!}\left(\frac{d}{dx}\right)^n (e^{-x} x^{n+k}) = \sum_{r=0}^{n} (-1)^r \binom{n+k}{n-r} \frac{x^r}{r!} \\ &= \sum_{r=0}^{n} (-1)^r \frac{(n+k)!}{(n-r)!(k+r)!} \frac{x^r}{r!} \end{aligned}$$

$$L_0^k(x) = 1 \qquad L_1^k(x) = -x + k + 1 \qquad L_2^k(x) = \frac{x^2}{2} - (k+2)x + \frac{(k+2)(k+1)}{2}$$

$$L_3^k(x) = -\frac{x^3}{6} + \frac{(k+3)x^2}{2} - \frac{(k+2)(k+3)x}{2} + \frac{(k+1)(k+2)(k+3)}{6}$$

Integral

$$\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx$$

$f(x)$	$F(x) = \int f(x) dx$
$\frac{1}{x^2 + a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\frac{x}{a}$
$\sqrt{a^2 - x^2}$	$\frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right)$
$\sqrt{x^2 \pm a^2}$	$\frac{1}{2}\left(x\sqrt{x^2 \pm a^2} \pm a^2 \log\left\lvert x + \sqrt{x^2 \pm a^2}\right\rvert\right)$
$\frac{1}{\sqrt{x^2 + a^2}}$	$\log\left\lvert x + \sqrt{x^2 + a^2}\right\rvert = \sinh^{-1}\frac{x}{a}$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\log\left\lvert x + \sqrt{x^2 - a^2}\right\rvert = \cosh^{-1}\frac{x}{a}$

Rational Function

$$f(x)=\frac{Q(x)}{P(x)} \quad \longrightarrow \quad f(x)=R(x)+\sum_{i=1}^{r}\sum_{k=1}^{m_i}\frac{A_{ik}}{(x-a_i)^k}+\sum_{j=1}^{s}\sum_{l=1}^{n_i}\frac{B_{jl}x+C_{jl}}{(x^2+\beta_j x+\gamma_j)^l}$$

: Partial fraction expansion

$$\int\frac{dx}{(x-\alpha)^n}=\begin{cases}\log|x-\alpha| & (n=1)\\ -\frac{1}{n-1}\frac{1}{(x-\alpha)^{n-1}} & (n>1)\end{cases}$$

$$\int\frac{Bx+C}{(x^2+\beta x+\gamma)^m}dx=\frac{B}{2}\int\frac{2x+\beta}{(x^2+\beta x+\gamma)^m}dx+C'\int\frac{dx}{(x^2+\beta x+\gamma)^m}$$

where $C'=C-B\frac{\beta}{2}$

$$\int\frac{2x+\beta}{(x^2+\beta x+\gamma)^m}dx=\begin{cases}\log(x^2+\beta x+\gamma) & (m=1)\\ -\frac{1}{m-1}\frac{1}{(x^2+\beta x+\gamma)^{m-1}} & (m>1)\end{cases}$$

$$\int\frac{dx}{(x^2+\beta x+\gamma)^m}=\int\frac{dt}{(t^2+\beta'^2)^m} \qquad t=x+\frac{\beta}{2} \qquad \beta'^2=\gamma-\frac{\beta^2}{4}$$

We define $I_n\equiv\int\frac{dx}{(x^2+a^2)^n}$

$$I_n=\frac{1}{2a^2(n-1)}\frac{x}{(x^2+a^2)^{n-1}}+\frac{2n-3}{a^2(2n-2)}\mathrm{I}_{n-1}$$

Rational Function of Trigonometric Functions

We let $P(X,Y)$ be a rational function of X,Y, and consider

$$\int P(\cos x,\sin x)dx$$

We define $t=\tan\frac{x}{2}$

$$\cos x=\frac{1-t^2}{1+t^2} \qquad \sin x=\frac{2t}{1+t^2} \qquad \frac{dt}{dx}=\frac{1+t^2}{2}$$

$$\int P(\cos x,\sin x)dx=\int P\left(\frac{1-t^2}{1+t^2},\frac{2t}{1+t^2}\right)\frac{2}{1+t^2}dt$$

which results in an integral of a rational function of t.

Sup.) $\qquad I_n=\int\sin^n x\,dx \qquad S_n=\int_0^{\frac{\pi}{2}}\sin^n x\,dx$

$$I_n=\frac{1}{n}(-\cos x\sin^{n-1}x+(n-1)I_{n-2}) \qquad I_0=x \qquad I_1=-\cos x$$

$$S_n=\begin{cases}\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2} & (n\geq 2\text{: even})\\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3} & (n\geq 3\text{: odd})\end{cases} \qquad S_0=\frac{\pi}{2} \qquad S_1=1$$

Rational Function of Quadratic Irrational Functions

$$\int P\left(x,\sqrt{a\,x^2+b\,x+c}\right)dx$$

i. Case of $a>0$

$$\sqrt{a\,x^2+b\,x+c}=t-\sqrt{a}x \qquad x=\frac{t^2-c}{2\sqrt{a}t+b}$$

The integrand goes to a rational function of t.

ii. Case of $a<0$

$$a\,x^2+b\,x+c=a(x-\alpha)(x-\beta) \qquad \alpha,\beta\in\Re$$

$$t=\sqrt{\frac{a(x-\alpha)}{x-\beta}}$$

The integrand goes to a rational function of t.

Change of Variables

$$\int\cdots\int_{\Omega} f(x_1\cdots x_n)dx_1\cdots dx_n=\int\cdots\int_{\Delta} f(\varphi_1(\xi_1\cdots\xi_n),\cdots,\varphi_n(\xi_1\cdots\xi_n))\left|\frac{\partial(\varphi_1\cdots\varphi_n)}{\partial(\xi_1\cdots\xi_n)}\right|d\xi_1\cdots d\xi_n$$

Other Integrals

$$\int_0^\infty e^{-ax^2}x^{2n}dx=\frac{(2n-1)!!}{2^{n+1}}\sqrt{\frac{\pi}{a^{2n+1}}} \qquad \int_0^\infty e^{-ax^2}dx=\frac{\sqrt{\pi}}{2\sqrt{a}}$$

$$\int_0^\infty e^{-ax^2}x^{2n+1}dx=\frac{n!}{2\,a^{n+1}}$$

First-Order Differential Equation

Variables-separable $\qquad y'=g(y)f(x)$

Equation	Change of Variables	New Equation
$y'=f(a\,x+b\,y+c)$	$u=a\,x+b\,y+c$	$\frac{du}{dx}=a+b\,f(u)$
$\frac{dy}{dx}=f\left(\frac{y}{x}\right)$ (Homogeneous)	$u=\frac{y}{x}$	$\frac{du}{dx}=\frac{f(u)-u}{x}$
$y'=f\left(\frac{a\,x+b\,y+c}{a'x+b'y+c'}\right)$ $(a\,b'-a'b\neq 0)$	$\begin{cases} a\alpha+b\beta+c=0 \\ a'\alpha+b'\beta+c'=0 \end{cases}$ $x=u+\alpha,\ y=v+\beta$	$\frac{dv}{du}=f\left(\frac{a+b\frac{v}{u}}{a'+b'\frac{v}{u}}\right)$
$(a\,b'-a'b=0)$	$^\exists\alpha$: $a=\alpha a',\ b=\alpha b'$	$y'=f\left(\alpha+\frac{c-\alpha c'}{a'x+b'y+c'}\right)$

First-Order Linear Differential Equation

$$y'+a(x)y=b(x)$$

$$p(x)=-\int_{x_0}^{x}a(x)dx \qquad u(x)=\int_{x_0}^{x}b(x)e^{-p(x)}dx$$

$$\Longrightarrow\ y(x)=u(x)e^{p(x)}+C\,e^{p(x)}$$

Sup.) An equation including derived functions of unknown y but not independent of variable x.

$$F(y,y',\cdots,y^{(n)})=0$$

If we let $y'=p$

$$y''=\frac{dp}{dy}p\ ,\ y'''=\frac{d^2p}{dy^2}p^2+\left(\frac{dp}{dy}\right)^2 p\ ,\ \ldots$$

which result in differential equations of unknown function p and independent variable y.

Linear Ordinary Differential Equation with Constant Coefficients

Homogeneous Equation

$$y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_2y''+a_1y'+a_0y=f(x)$$

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_2D^2 + a_1D + a_0)y = f(x) \qquad \text{where } D \equiv \frac{d}{dx}$$

$$\mathrm{P}(D)y = f(x)$$

Homogeneous equation

$$\mathrm{P}(D)y = 0$$

$$\alpha_1 \cdots \alpha_n \in C : \qquad \text{Solution of } \mathrm{P}(t) = 0$$

$$(D - \alpha_1)(D - \alpha_2) \cdots (D - \alpha_n)y = 0$$

Fundamental solutions

$$(D - \alpha_i)y = 0 \qquad \Longrightarrow \qquad y = C_i\, e^{\alpha_i x}$$

$$(D - \alpha_j)^m y = 0 \qquad \Longrightarrow \qquad y = (C_0^j + C_1^j x + \cdots + C_{m-1}^j x^{m-1})e^{\alpha_j x}$$

(Let $y = e^{\alpha x} u$)

Method of Variation of Parameters

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

If fundamental solutions of its homogeneous equation $y_1\, y_2 \cdots y_n$ are known,

Wronskian $$W(y_1, \cdots, y_n) \equiv \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Green function $$G(x, s) \equiv \frac{1}{W(y_1(s), \cdots, y_n(s))} \begin{vmatrix} y_1(s) & \cdots & y_n(s) \\ \cdots & \cdots & \cdots \\ y_1^{(n-2)}(s) & \cdots & y_n^{(n-2)}(s) \\ y_1(x) & \cdots & y_n(x) \end{vmatrix}$$

The solution of the non-homogeneous equation is

$$y(x) = \sum_{j=0}^{n-1} c_j y_j(x) + \int_{x_0}^{x} G(x, s) f(s) ds$$

Lagrange's Differential Equation

$$p(x, y, u)\frac{\partial u}{\partial x} + q(x, y, u)\frac{\partial u}{\partial y} = r(x, y, u)$$

Auxiliary equation $$\frac{dx}{p(x, y, u)} = \frac{dy}{q(x, y, u)} = \frac{du}{r(x, y, u)}$$

We let two independent solutions of this equation be

$$g(x, y, u) = a \qquad\qquad h(x, y, u,) = b$$

General solutions of Lagrange's equation are any functions of g and h.

$$f(g, h) = 0$$

Laplace Equation in 2 Dimension

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\xi = x + i\, y \qquad \eta = x - i\, y \qquad\qquad \phi(x, y) = \tilde{\phi}(\xi, \eta)$$

$$\boldsymbol{\nabla}^2 \phi = 4\frac{\partial^2}{\partial \xi \partial \eta}\tilde{\phi} = 0$$

General solutions $\tilde{\phi}(\xi,\eta) = f_1(\xi) + f_2(\eta)$

Rotating symmetry (Function of $\xi\eta = x^2 + y^2$) $g(\xi\eta) = a\ln\xi\eta + b$

Fourier Transform

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk$$

General Formulas in Fourier Transform

$$\int_{-\infty}^{\infty} \hat{f}(k)\hat{g}^*(k)dk = \int_{-\infty}^{\infty} f(x)g^*(x)dx \qquad \text{(Parseval's identity)}$$

$f(x)$	$\longleftrightarrow$	$\hat{f}(k)$
$f^*(x)$		$\hat{f}^*(-k)$
$\hat{f}(x)$		$f(-k)$
$f\left(\frac{x}{a} + b\right)$		$\lvert a\rvert e^{iabk}\hat{f}(a\,k)$
$f(a\,x)e^{-ibx}$		$\frac{1}{\lvert a\rvert}\hat{f}\left(\frac{k+b}{a}\right)$
$x^n f(x)$		$(i)^n\frac{d^n\hat{f}(k)}{dk^n}$
$f^{(n)}(x)$		$(i\,k)^n\hat{f}(k)$
$(f*g)(x)$		$\hat{f}(k)\hat{g}(k)$
$f(x)g(x)$		$(\hat{f}*\hat{g})(k)$

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-t)g(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(x-t)dt$$

Examples of Fourier Transform

$f(x)$	$\longleftrightarrow$	$\hat{f}(k)$	
$\frac{1}{x^2+a^2}$	$(a>0)$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert k\rvert}}{a}$	
$\frac{1}{\sqrt{x^2+a^2}}$		$\sqrt{\frac{2}{\pi}}K_0(a\lvert k\rvert)$	(Modified Bessel)
e^{-ax^2}		$\frac{1}{\sqrt{2a}}e^{-\frac{k^2}{4a}}$	
$\tan^{-1}\frac{x}{a}$		$-i\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert k\rvert}}{k}$	
$\delta(x)$		$\frac{1}{\sqrt{2\pi}}$	
$\frac{1}{x}$		$-i\sqrt{\frac{\pi}{2}}\mathrm{sgn}(k)$	
$\theta(x)$	Heviside step function	$\sqrt{\frac{\pi}{2}}\delta(k) - \frac{i}{\sqrt{2\pi}k} = \frac{1}{i\sqrt{2\pi}}\frac{1}{k-i\epsilon}$	

$$e^{-ax}\theta(x) \qquad \frac{1}{\sqrt{2\pi}(a+ik)}$$

$$\sqcap(x) \equiv \theta\left(x+\frac{1}{2}\right) - \theta\left(x-\frac{1}{2}\right) \qquad \sqrt{\frac{2}{\pi}}\frac{\sin\left(\frac{k}{2}\right)}{k}$$

Delta Functions

$$\delta(x) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} dk$$

$$\delta(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ikx} \qquad (-2\pi < x < 2\pi) \qquad \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ikx} = \sum_{k=-\infty}^{\infty} \delta(x-2\pi k)$$

$$\delta(a\,x) = \frac{\delta(x)}{|a|} \qquad x\delta(x) = 0 \qquad x\delta'(x) = -\delta(x)$$

$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|} \qquad f(x_i) = 0$$

$$\lim_{\epsilon \to +0} \frac{1}{x \pm i\epsilon} = V.P.\frac{1}{x} \mp i\pi\delta(x)$$

Integral Equation

Abel's Problem

$$f(x) = \int_{-\infty}^{\infty} K(x-y)u(y)dy \qquad u(x)\text{: Unknown function}$$

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{f}(k)}{\hat{K}(k)} e^{ikx} dk$$

$K(x-y)$: Kernel

$$f(x) = g(x) + \lambda\int_{-\infty}^{\infty} K(x-y)f(y)dy \qquad f(x)\text{: unknown function}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\hat{g}(k)}{1-\sqrt{2\pi}\lambda\hat{K}(k)} e^{ikx} dk$$

Green Functions

Consider a partial differential equation of type $Lu(x) = f(x)$.

$$L = F(D_{x_1}, \cdots, D_{x_n}) \equiv F(D_{x_i})$$

$$F(D_{x_i})G(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}') \qquad \boldsymbol{r} = (x_1, \cdots, x_n)$$

Then

$$G(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{(2\pi)^n}\int \frac{e^{ik(\boldsymbol{r}-\boldsymbol{r}')}}{F(i\,k_i)} d\boldsymbol{k}$$

Diffusion Equation in 1 Dimension

$$L_{x,t} = D_x^2 - D_t$$

$$G(x-x', t-t') = \begin{cases} -\dfrac{1}{\sqrt{4\pi|t-t'|}}\exp\left\{-\dfrac{(x-x')^2}{4|t-t'|}\right\} & (t > t') \\ 0 & (t < t') \end{cases}$$

Initial-value problem

$$(D_x^2 - D_t)u(x,t) = 0 \qquad \text{with } u(x,0) = f(x)$$

$$(D_x^2 - D_t)u(x,t) = -\delta(t)f(x)$$

Wave Equation in 1 Dimension

$$L_{x,t} = D_x^2 - D_t^2$$

$$G(x-x', t-t') = \begin{cases} -\dfrac{1}{2}\theta(t-t'-|x-x'|) & (t > t') \\ 0 & (t < t') \end{cases}$$

Laplace's Equation in 2 Dimension

$$L_{x,y} = D_x^2 + D_y^2$$

$$G(\boldsymbol{r}-\boldsymbol{r}') = \frac{1}{2\pi}\ln(|\boldsymbol{r}-\boldsymbol{r}'|)$$

Sup.) Poisson integral formula (Dirichlet problem)

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(R_0^2 - r^2)u(R_0,\phi)}{R_0^2 - 2R_0 r\cos(\theta-\phi) + r^2}d\phi$$

Cauchy integral formula

$$\boxed{f(z) = \frac{1}{2\pi i}\int_c \frac{f(\zeta)}{\eta - z}d\zeta}$$

Laplace's Equation in 3 Dimension

$$L_{x,y,z} = D_x^2 + D_y^2 + D_z^2$$

$$G(\boldsymbol{r}-\boldsymbol{r}') = -\frac{1}{4\pi}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

Diffusion Equation in 3 Dimension

$$L_{x,y,z,t} = D_x^2 + D_y^2 + D_z^2 - D_t$$

$$G(\boldsymbol{r}-\boldsymbol{r}', t-t') = -\frac{1}{[4\pi(t-t')]^{3/2}}\exp\left[-\frac{(\boldsymbol{r}-\boldsymbol{r}')^2}{4|t-t'|}\right] \qquad (t > t')$$

Wave Equation in 3 Dimension

$$L_{x,y,z,t} = D_x^2 + D_y^2 + D_z^2 - D_t^2,$$

$$G(\boldsymbol{r}-\boldsymbol{r}', t-t') = \begin{cases} -\dfrac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\delta(t-t'-|\boldsymbol{r}-\boldsymbol{r}'|) & (t > t') \\ 0 & (t < t') \end{cases}$$

Sup.) Consider $L_{\boldsymbol{r},t} = F(\Delta, D_t)$. We let Green function of $L_{x,t} = F(D_x^2, D_t)$ be $g_1(x,t)$. Then Green function of $L_{\boldsymbol{r},t}$ is

$$G(\boldsymbol{r},t) = -\frac{1}{2\pi r}D_r g_1(r,t) \qquad r = |\boldsymbol{r}|$$

Laplace Transform

$$\tilde{f}(s) \equiv \int_0^\infty f(x)e^{-st}dx \qquad\qquad f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\tilde{f}(s)ds$$

$c > \sigma$: Region of convergence

$f(t)$	$\longleftrightarrow$	$\tilde{f}(s)$
$f(\alpha t)$		$\dfrac{1}{\alpha}\tilde{f}\left(\dfrac{s}{\alpha}\right) \qquad (\alpha > 0)$
$\begin{cases} f(t-a) & (x \geq a) \\ 0 & (x < a) \end{cases}$		$e^{-sa}\tilde{f}(s)$
$f(t)e^{bt}$		$\tilde{f}(s-b) \qquad (s > \sigma + b)$

$t^n f(t)$	$\left(-\frac{d}{ds}\right)^n \tilde{f}(s)$
$\frac{1}{t^n} f(t)$	$\int_s^\infty \int_{\sigma_{n-1}}^\infty \cdots \int_{\sigma_1}^\infty \tilde{f}(\sigma) d\sigma d\sigma_1 \cdots d\sigma_{n-1}$
$f'(t)$	$s\,\tilde{f}(s) - f(0)$
$f''(t)$	$s^2\,\tilde{f}(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^n \tilde{f}(s) - \sum_{r=0}^{n-1} s^{n-r-1} f^{(r)}(0)$
$\int_0^t \int_0^{\tau_{n-1}} \cdots \int_0^{\tau_1} f(\tau) d\tau d\tau_1 \cdots d\tau_{n-1}$	$\frac{1}{s^n} \tilde{f}(s)$
$t^m f^{(n)}(t) \qquad (m \geq n)$	$\left(-\frac{d}{ds}\right)^m [s^n \tilde{f}(s)]$
$\int_0^t f(\tau) g(t-\tau) d\tau \equiv (f * g)(t)$	$\tilde{f}(s)\,\tilde{g}(s)$

$$\int_0^t f(\tau) g(t-\tau) d\tau = \int_0^t f(t-\tau) g(\tau) d\tau$$

Examples of Laplace Transform

$f(t)$	$\longleftrightarrow$	$\tilde{f}(s)$	
$\delta(t)$		1	
$\theta(t)$	Heviside step function	$\frac{1}{s}$	
t^ν	$(-1 < \nu < \infty)$	$\frac{\Gamma(\nu+1)}{s^{\nu+1}}$	$n! = \Gamma(n+1)$
e^{at}		$\frac{1}{s-a}$	
$e^{at} t^{\nu-1}$	$(\nu > 0)$	$\frac{\Gamma(\nu)}{(s-a)^\nu}$	
$\sin(at)$		$\frac{a}{s^2+a^2}$	
$\cos(at)$		$\frac{s}{s^2+a^2}$	
$J_0(2\sqrt{\lambda t})$		$\frac{1}{s} e^{-\lambda/s}$	