Commutation Relation

$[\hat{A},\hat{B}] = -[\hat{B},\hat{A}]$ $\qquad$ $[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$

$[\hat{A},[\hat{B},\hat{C}]] + [\hat{B},[\hat{C},\hat{A}]] + [\hat{C},[\hat{A},\hat{B}]] \equiv 0$

$e^{\hat{B}}\hat{A}e^{-\hat{B}} = \hat{A} + [\hat{B},\hat{A}] + \cdots + \frac{1}{n!}[\hat{B},[\hat{B},\cdots,[\hat{B},\hat{A}]\cdots]] + \cdots$ $\qquad$ $e^{\hat{B}}F(\hat{A})e^{-\hat{B}} = F(e^{\hat{B}}\hat{A}e^{-\hat{B}})$

$[\hat{A},\hat{B}] = \text{C-num} \implies [F(\hat{A}),\hat{B}] = F'(\hat{A})[\hat{A},\hat{B}]$ $\qquad$ $e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}$

$\Delta A \Delta B \geqq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$ $\qquad$ where $\Delta A \equiv \sqrt{\langle(\hat{A}-\langle\hat{A}\rangle)^2\rangle} = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}$

$[\hat{p},\hat{x}] = -i\hbar$

$h = 2\pi\hbar$: phase space volume for one quantum state in 1-dim.

Minimum Wave Packet: $\Delta A \Delta B = \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

$[\langle[\hat{A},\hat{B}]\rangle(\hat{A}-\langle\hat{A}\rangle) - 2(\Delta A)^2(\hat{B}-\langle\hat{B}\rangle)]|\psi\rangle = 0$

$\hat{A} = \hat{x},\ \hat{B} = \hat{p} \implies \psi = \text{Const} \times \exp\left[\frac{i}{\hbar}\langle\hat{p}\rangle(\hat{x}-\langle\hat{x}\rangle)\right]\exp\left[-\frac{(\hat{x}-\langle\hat{x}\rangle)^2}{(2\Delta x)^2}\right]$

Ket $|\psi\rangle$

$\psi(x,t) = \langle x|\psi\rangle$ $\qquad$ $\langle x|x'\rangle = \delta(x-x')$ $\qquad$ $\sum_x |x\rangle\langle x| = 1$

Expansion with Eigen Ket $\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle$

$$\begin{aligned}|\psi,t\rangle &= \sum_n |\phi_n\rangle\langle\phi_n|e^{-\frac{i}{\hbar}E_n(t-t')}|\psi,t'\rangle \\ &= \sum_{n,x,x'} |x\rangle\phi_n(x)\phi_n^*(x')e^{-\frac{i}{\hbar}E_n(t-t')}\langle x'|\psi,t'\rangle \\ &= \sum_{n,x,x'} |x\rangle G(x,t;x',t')\langle x'|\psi,t'\rangle\end{aligned}$$

$\langle x|\hat{A}|\psi\rangle = \sum_{x'} \langle x|\hat{A}|x'\rangle\langle x'|\psi\rangle = \sum_{x'} A_{xx'}\psi(x')$

$\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$ $\qquad$ Expected value

Schrödinger Equation

$\hat{\boldsymbol{p}} = -i\hbar\boldsymbol{\nabla}$ $\qquad$ $\hat{\boldsymbol{x}} = \boldsymbol{x}$ $\qquad$ Coordinate representation

$\hat{H}|\psi,t\rangle = i\hbar\frac{\partial}{\partial t}|\psi,t\rangle$ $\qquad$ $\left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + U(\boldsymbol{x})\right]\psi(\boldsymbol{x},t) = i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{x},t)$

$\psi(\boldsymbol{x},t) = \frac{1}{(2\pi\hbar)^{3/2}}e^{\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{x}-Et)}$ $\qquad$ Plane wave

$\hat{\boldsymbol{p}} \to \hat{\boldsymbol{p}} - q\boldsymbol{A} = -i\hbar\boldsymbol{\nabla} - q\boldsymbol{A}$ $\qquad$ with vector potential $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$

Heisenberg Representation

$\hat{U}(t,t') = e^{-\frac{i}{\hbar}\hat{H}(t-t')}$ $\qquad$ Evolution operator

Time Evolution of Eigen State $\qquad$ $\hat{U}$: Unitary $\qquad$ $\hat{U}^\dagger = \hat{U}^{-1}$

$\hat{U}|\phi_n\rangle = e^{-\frac{i}{\hbar}E_nt}|\phi_n\rangle$

$\hat{O}_{\mathrm{S}}$: Schrödinger representation $\qquad$ $\hat{O}_{\mathrm{H}}$: Heisenberg representation

$|\psi\rangle_{\rm H} = |\psi, t_0\rangle_{\rm H} \equiv |\psi(t=t_0)\rangle_{\rm S} = \hat{U}^\dagger(t,t_0)|\psi(t)\rangle_{\rm S}$: State

$$\hat{O}_{\rm H}(t) = \hat{U}^{-1}(t)\hat{O}_{\rm S}\hat{U}(t) \qquad i\hbar\frac{\partial}{\partial t}\hat{O}_{\rm H}(t) = [\hat{O}_{\rm H}(t), \hat{H}]$$

$$[\hat{A}_{\rm S}, \hat{B}_{\rm S}] = \hat{C}_{\rm S} \quad \Longrightarrow \quad [\hat{A}_{\rm H}, \hat{B}_{\rm H}] = \hat{C}_{\rm H}$$

$$\hat{H}_{\rm H} = \frac{\hat{\boldsymbol{p}}_{\rm H}^2}{2m} + V(\hat{\boldsymbol{r}}_{\rm H}) \quad \Longrightarrow \quad \frac{d}{dt}\hat{\boldsymbol{r}}_{\rm H} = \frac{\hat{\boldsymbol{p}}_{\rm H}}{m} \qquad \frac{d}{dt}\hat{\boldsymbol{p}}_{\rm H} = -\frac{\partial}{\partial \hat{\boldsymbol{r}}_{\rm H}}V(\hat{\boldsymbol{r}}_{\rm H})$$

Wave Function in Heisenberg Representation

$\hat{x}_{\rm H}(t)|x,t\rangle_{\rm H} = x|x,t\rangle_{\rm H}$, $\hat{x}|x\rangle_{\rm S} = x|x\rangle_{\rm S}$: Coordinate eigen ket at time t

$${}_{\rm H}\langle x,t|\psi\rangle_{\rm H} = {}_{\rm S}\langle x|\psi,t\rangle_{\rm S}$$

Interaction Representation

$$\hat{H} = \hat{H}_0 + \hat{V}$$

$$\hat{O}_{\rm I} \equiv e^{\frac{i}{\hbar}\hat{H}_0 t}\,\hat{O}_{\rm S}\,e^{-\frac{i}{\hbar}\hat{H}_0 t} \qquad |\psi(t)\rangle_{\rm I} \equiv e^{\frac{i}{\hbar}\hat{H}_0 t}|\psi(t)\rangle_{\rm S}$$

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle_{\rm I} \equiv \hat{V}_{\rm I}|\psi(t)\rangle_{\rm I} \qquad \hat{V}_{\rm I} \equiv e^{\frac{i}{\hbar}\hat{H}_0 t}\,\hat{V}_{\rm S}\,e^{-\frac{i}{\hbar}\hat{H}_0 t}$$

$$i\hbar\frac{\partial}{\partial t}\hat{O}_{\rm I}(t) = [\hat{O}_{\rm I}(t), \hat{H}_0]$$

$$[\hat{A}_{\rm S}, \hat{B}_{\rm S}] = \hat{C}_{\rm S} \quad \Longrightarrow \quad [\hat{A}_{\rm I}, \hat{B}_{\rm I}] = \hat{C}_{\rm I}$$

$$\hat{H}_0 = \frac{\hat{\boldsymbol{p}}_{\rm I}^2}{2m} + V_0(\hat{\boldsymbol{r}}_{\rm I}) \quad \Longrightarrow \quad \frac{d}{dt}\hat{\boldsymbol{r}}_{\rm I} = \frac{\hat{\boldsymbol{p}}_{\rm I}}{m} \qquad \frac{d}{dt}\hat{\boldsymbol{p}}_{\rm I} = -\frac{\partial}{\partial \hat{r}_{\rm I}}V_0(\hat{\boldsymbol{r}}_{\rm I})$$

Evolution Operator

$$\hat{U}(t'',t') = \exp\left\{-\frac{i}{\hbar}\hat{H}(t''-t')\right\}$$

Ψ': A state in Schrödinger picture at t'. Ψ'': The state where Ψ' will turn out to be at t''.

$$\Psi'' = U(t'',t')\Psi'$$

Matrix elements: $\langle q''|U(t'',t')|q'\rangle = \langle q'',t''|q',t'\rangle$

Probability Flux

$$\boldsymbol{j}(\boldsymbol{x},t) = -\frac{i\hbar}{2m}(\psi^*\nabla\psi - (\nabla\psi^*)\psi) = \frac{\hbar}{m}\mathrm{Im}(\psi^*\nabla\psi) \qquad \psi = \sqrt{\rho(\boldsymbol{x},t)}\,e^{\frac{i}{\hbar}S(\boldsymbol{x},t)} \;\Rightarrow\; \boldsymbol{j} = \frac{\rho}{m}\nabla S$$

$\rho(\boldsymbol{x},t) = \psi^*\psi$ $\qquad \dfrac{\partial\rho}{\partial t} + \nabla\boldsymbol{j} = 0$: Equation of continuity

Harmonic Oscillator

1-Dimensional Harmonic Oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \qquad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right) \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right) \qquad [\hat{a},\hat{a}^\dagger] = 1$$

$$[\hat{a}^\dagger\hat{a}, \hat{a}] = -\hat{a} \qquad [\hat{a}^\dagger\hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \qquad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \qquad |n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle$$

Eigen Function

$$\hat{a}^\dagger = -\frac{i}{\sqrt{2m\hbar\omega}}\exp\left(\frac{1}{2}\alpha^2\hat{x}^2\right)\hat{p}\exp\left(-\frac{1}{2}\alpha^2\hat{x}^2\right) \qquad \alpha \equiv \sqrt{\frac{m\omega}{\hbar}}$$

$$\psi_n(x) = \sqrt{\frac{\alpha}{n!2^n\sqrt{\pi}}}H_n(\alpha x)\exp\left(-\frac{1}{2}\alpha^2\hat{x}^2\right)$$

$$H_n(z) \equiv (-1)^n\exp(z^2)\left(\frac{d}{dx}\right)^n\exp(-z^2) : \text{ Hermite polynomials}$$

Coherent State

$$\hat{a}|a\rangle = a|a\rangle \qquad a \in \mathbb{C}$$

$$|a\rangle = \exp(a\hat{a}^\dagger - a^*\hat{a})|0\rangle = \exp\left(-\frac{1}{2}|a|^2\right)\sum_{n=0}^{\infty}\frac{a^n}{\sqrt{n!}}|n\rangle \qquad \langle a|a'\rangle = \exp\left(-\frac{1}{2}|a-a'|^2\right)$$

$$\int da|a\rangle\langle a| = \pi : \text{ Over complete}$$

$$a\hat{a}^\dagger - a^*\hat{a} = \frac{i}{\hbar}(p\hat{x} - x\hat{p}) \qquad \text{where } a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right)$$

$$\langle a|\hat{x}|a\rangle = x \qquad \langle a|\hat{p}|a\rangle = p$$

$$\Delta x^2 \equiv \langle a|(\hat{x} - \langle\hat{x}\rangle)^2|a\rangle = \langle 0|\hat{x}^2|0\rangle = \frac{1}{2\alpha^2}$$

$$\Delta p^2 \equiv \langle a|(\hat{p} - \langle\hat{p}\rangle)^2|a\rangle = \langle 0|\hat{p}^2|0\rangle = \frac{\hbar^2\alpha^2}{2} \qquad \alpha \equiv \frac{m\omega}{\hbar}$$

Time Evolution of Coherent State

$$|a(t)\rangle = \exp\left(-\frac{i}{2}\omega t\right)\exp[a(t)\hat{a}^\dagger - a^*(t)\hat{a}]|0\rangle \qquad a(t) = a\,e^{-i\omega t}$$

$$\begin{aligned} x(t) \equiv \langle a(t)|\hat{x}|a(t)\rangle &= \sqrt{\frac{\hbar}{2m\omega}}(a^*(t) + a(t)) \\ &= \sqrt{\frac{2\hbar}{m\omega}}|a|\cos(\omega t + \delta) \end{aligned}$$

$$\begin{aligned} p(t) \equiv \langle a(t)|\hat{p}|a(t)\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}(a^*(t) - a(t)) \\ &= -\sqrt{2\hbar m\omega}|a|\sin(\omega t + \delta) \end{aligned}$$

Orbital Angular Momentum of Particle

$$\hbar\hat{\boldsymbol{L}} = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}} \qquad [\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k$$

$$[\hat{L}_\pm, \hat{L}_z] = \mp\hat{L}_\pm \qquad [\hat{L}^2, \hat{L}_z] = [\hat{L}^2, \hat{L}_\pm] = 0 \qquad \hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$$

$$\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_z(\hat{L}_z + 1) \qquad \hat{L}_+\hat{L}_- = \hat{L}^2 - \hat{L}_z(\hat{L}_z - 1)$$

$$\hat{L}_x = -i\left(-\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right) \qquad \hat{L}_y = -i\left(\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$$

$$\hat{L}_+ = e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) \qquad \hat{L}_- = e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

$$\hat{L}_z = -i\frac{\partial}{\partial\varphi} \qquad \hat{\boldsymbol{L}}^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial\varphi}\right)^2$$

Eigen State

$$\hat{L}^2|l,m\rangle = l(l+1)|l,m\rangle \qquad \hat{L}_z|l,m\rangle = m|l,m\rangle$$

Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + U(|\boldsymbol{r}|) = -\frac{\hbar^2}{2m}\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2 r + \frac{\hbar^2\hat{\boldsymbol{L}}^2}{2m\,r^2} + U(|\boldsymbol{r}|)$$

If we let $\psi(r,\theta,\varphi) = \frac{u(r)}{r}Y_{lm}(\theta,\varphi)$

$$\left[-\frac{\hbar^2}{2m}\left(\frac{d}{dr}\right)^2 + \frac{\hbar^2 l(l+1)}{2m\,r^2} + U(|\boldsymbol{r}|)\right]u(r) = E\,u(r)$$

$$Y_{lm}(\theta,\varphi) \equiv \frac{1}{\sqrt{2\pi}}e^{im\varphi}(-1)^{\frac{m+|m|}{2}}\sqrt{\frac{(2l+1)(l-|m|)!}{2(l+|m|)!}}P_l^{|m|}(\cos\theta)$$

$$\text{where}\quad P_l^m(x) = \frac{(-1)^l}{2^l l!}(1-x^2)^{m/2}\left(\frac{d}{dx}\right)^{l+m}(1-x^2)^l$$

$$\hat{L}_{\pm}|l,m\rangle = \sqrt{(l\mp m)(l\pm m+1)}\,|l,m\pm 1\rangle$$

Spin Angular Momentum

Pauli Spin Matrix

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k = \overset{i\downarrow j\rightarrow}{\begin{pmatrix} 1 & i\sigma_3 & -i\sigma_2 \\ -i\sigma_3 & 1 & i\sigma_1 \\ i\sigma_2 & -i\sigma_1 & 1 \end{pmatrix}} \qquad \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b}) = \boldsymbol{a}\cdot\boldsymbol{b} + i\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b}) \qquad (\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 = \boldsymbol{p}^2$$

$$\{\boldsymbol{\sigma}\cdot(-i\boldsymbol{\nabla} - q\boldsymbol{A})\}\{\boldsymbol{\sigma}\cdot(-i\boldsymbol{\nabla} - q\boldsymbol{A})\}\varphi = (-i\boldsymbol{\nabla} - q\boldsymbol{A})^2\varphi - q\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A})\varphi$$

$$\exp(i\theta\sigma_i) = \cos\theta + i\sigma_i\sin\theta$$

$$a + \sigma_i a \sigma_i = 2\mathrm{Tr}(a) I_2$$

Composition of Angular Momentum

Clebsch-Gordan Coefficient

j_1, j_2 : given

$$|j_1 j_2; m_1 m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle \quad \text{or} \quad |j_1 j_2; j\,m\rangle$$

$$|j_1 j_2; j\,m\rangle = \sum_{m_1,m_2}\langle j_1 j_2; m_1 m_2 | j_1 j_2; j\,m\rangle |j_1, m_1\rangle \otimes |j_2, m_2\rangle$$

$$\langle j_1 j_2; m_1 m_2 | j_1 j_2; j\,m\rangle \text{ : CG coefficient}$$

$$|j_1 j_2; j\,m\rangle = \left[\frac{(j+m)!}{(2j)!(j-m)!}\right]^{1/2}\hat{J}_-^{j-m}|j_1 j_2; j\,j\rangle$$

Racah's formula

$$\begin{aligned}\langle j_1 j_2; m_1 m_2 | j_1 j_2; j\,m\rangle &= \sqrt{2j+1}\left[\frac{(-j+j_1+j_2)!\cdot(j+j_1-j_2)!\cdot(j-j_1+j_2)!}{(j+j_1+j_2+1)!}\right]^{1/2} \\ &\times[(j_1+m_1)!\cdot(j_1-m_1)!\cdot(j_2+m_2)!\cdot(j_2-m_2)!\cdot(j_2+m_2)!\cdot(j_2-m_2)!]^{1/2} \\ &\times\sum_k\frac{(-1)^k}{k!}[(j_1+j_2-j-k)!\cdot(j_1-m_1-k)!\cdot(j_2+m_2-k)! \\ &\qquad\cdot(j-j_2+m_1+k)!\cdot(j-j_1-m_2+k)!]^{-1}\end{aligned}$$

d-function

$$d^j_{n,m}(\theta) \equiv \langle j, n|e^{-i\theta\hat{J}_y}|j, m\rangle$$

$$d^j_{n,m}(\theta) = d^j_{-m,-n}(\theta) = (-1)^{n-m} d^j_{m,n}(\theta) \qquad d^j_{n,m}(\theta) = (-1)^{j+n} d^j_{n,-m}(\pi-\theta)$$

$$\begin{pmatrix} d^{1/2}_{1/2,1/2} & d^{1/2}_{1/2,-1/2} \\ d^{1/2}_{-1/2,1/2} & d^{1/2}_{-1/2,-1/2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$\begin{pmatrix} d^1_{1,1} & d^1_{1,0} & d^1_{1,-1} \\ d^1_{0,1} & d^1_{0,0} & d^1_{0,-1} \\ d^1_{-1,1} & d^1_{-1,0} & d^1_{-1,-1} \end{pmatrix} = \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & -\cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix}$$

Perturbation Theory $(\hbar = 1)$

$$H_0|n\rangle = i\frac{\partial}{\partial t}|n\rangle \qquad |n,t\rangle = U_0(t,t_0)|n,t_0\rangle = e^{-iE_n(t-t_0)}|n,t_0\rangle$$

$$(H_0+V)|\psi\rangle = i\frac{\partial}{\partial t}|\psi\rangle \qquad |\psi,t\rangle = U(t,t_0)|\psi,t_0\rangle$$

Integral equation on $U(t,t_0)$

$$U(t,t_0) = U_0(t,t_0) - i\int_{t_0}^{t} dt' U_0(t,t')V(t')U(t',t_0)$$

Perturbation Expansion $\quad U(t,t_0) = U_0(t,t_0) + U^{(1)}(t,t_0) + U^{(2)}(t,t_0) + \cdots$

$$U^{(1)}(t,t_0) = -i\int_{t_0}^{t} dt' U_0(t,t')V(t')U_0(t',t_0)$$

$$\begin{aligned} U^{(2)}(t,t_0) &= \frac{(-i)^2}{2!}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' U_0(t,t')V(t')U_0(t',t'')V(t'')U_0(t'',t_0) \\ &= \frac{(-i)^2}{2!}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' T[U_0(t,t')V(t')U_0(t',t'')V(t'')U_0(t,t_0)] \end{aligned}$$

$$U(t,t_0) = U_0(t,t_0) T\exp\left[-i\int_{t_0}^{t} dt' U_0(t_0,t')\, V(t') U_0(t',t_0)\right]$$

where $\mathcal{T}$ is time ordering operator (T-product)

Transition Amplitude

$$A^{(1)}_{fi} \equiv \langle f,t|U^{(1)}(t,t_0)|i,t_0\rangle = -i\int_{t_0}^{t} dt' V_{fi}(t') e^{i(E_f-E_i)(t'-t_0)}$$

$$V_{fi}(t') \equiv \langle f,t_0|V(t')|i,t_0\rangle = \int dx' \phi^*_f(x',t_0)V(x',t')\phi_i(x',t_0)$$

$$A^{(2)}_{fi} = (-i)^2 \sum_n \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' V_{fn}(t') e^{i(E_f-E_n)(t'-t_0)} V_{ni}(t'') e^{i(E_n-E_i)(t''-t_0)}$$

In case that V is independent of t

$$t_0 \to -\infty \qquad t \to +\infty$$

$$T_{fi} \equiv \langle f,\infty|U(\infty,-\infty)|i,-\infty\rangle$$

$$T^{(1)}_{fi} = -2\pi i V_{fi}\,\delta(E_f - E_i)$$

$$T^{(2)}_{fi} = -2\pi i \sum_n V_{fn} \frac{1}{E_i - E_n + i\varepsilon} V_{ni}\,\delta(E_f - E_i)$$

Fermi Golden Rule

$$W_{i\to f} = \frac{2\pi}{\hbar}|M_{fi}|^2\rho \qquad M_{fi} = V_{fi} + \sum_{n\neq i}\frac{V_{fn}V_{ni}}{E_i - E_n} + \cdots \qquad V_{fi} \equiv \langle f|\hat{V}|i\rangle$$

ρ: Number of the final state per unit energy $\dfrac{dn}{dE} = \delta(E_f - E_i)\dfrac{Vd^3p}{(2\pi\hbar)^3}$