Notation

$$E(x) \equiv \lim_{\nu\to\infty}\left(\sum_{i=1}^{\nu} x_i/\nu\right) \qquad V(x) = E(x^2) - E^2(x)$$

$$E(a\,x + b\,y) = a\,E(x) + b\,E(y) \qquad V(a\,x + b) = a^2 V(x)$$

$x,\ y$: independant: $\quad E(x\,y) = E(x)E(y) \quad V(x+y) = V(x) + V(y)$

x_i: Element in population $\quad \bar{x}$: Mean of population $\quad s$: Standard deviation of population

$$\bar{x} = E(x) \qquad s^2 = E\{(x-\bar{x})^2\} = E(x^2) - \bar{x}^2$$

$\bar{X} \equiv \frac{1}{N}\sum_{i=1}^{N} X_i$: Sample mean $\quad E(\bar{X}) = \bar{x}$ $\quad$ N: Sample size; X_i: i-th element in sample

$S^2 \equiv \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})^2 = \frac{1}{N-1}\left(\sum_{i=1}^{N} X_i^2 - N\bar{X}^2\right)$: Sample variance $\quad E(S^2) = s^2$

$$E(X_i) = \bar{x} \qquad E(X_i^2) = E(x^2)$$

$$\begin{aligned} E(X_i\bar{X}) &= \frac{1}{N}E\left\{X_i\left(X_i + \sum_{k\neq i} X_k\right)\right\} = \frac{1}{N}\left\{E(X_i^2) + \sum_{k\neq i} E(X_i X_k)\right\} \\ &= \frac{1}{N}E(x^2) + \frac{N-1}{N}\bar{x}^2 \end{aligned}$$

$$E(\bar{X}\cdot\bar{X}) = E\left(\frac{1}{N}\sum_{k} X_k\bar{X}\right) = E(X_k\bar{X})$$

$$E\{(X_i - \bar{X})^2\} = E(X_i^2 - 2X_i\bar{X} + \bar{X}^2) = \frac{N-1}{N}\{E(x^2) - \bar{x}^2\} = \frac{N-1}{N}s^2$$

Covariance and Correlation Coefficient of xy

$$\sigma_{xy} \equiv \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{N}\left\{\left(\sum_{i=1}^{N} x_i\,y_i\right) - N\bar{x}\bar{y}\right\} \qquad \rho_{xy} \equiv \frac{\sigma_{xy}}{s_x s_y}$$

Change of Variables

Change of random variable $\boldsymbol{D} \subseteq \Re^n \to \boldsymbol{D'} \subseteq \Re^n$

$$X_i \in \boldsymbol{D} \longmapsto Y_i = f_i(X_1, \cdots, X_n) \in \boldsymbol{D'}$$

Relation between probability density function $p(X_1, \cdots, X_n)$ and $q(Y_1, \cdots, Y_n)$

$$p(X_1, \cdots, X_n) = \left|\frac{\partial(Y_1, \cdots, Y_n)}{\partial(X_1, \cdots, X_n)}\right| q(Y_1, \cdots, Y_n)$$

where we let the determinant non-zero everywhere $\longrightarrow$ One-to-one between $\boldsymbol{D}$ and $\boldsymbol{D'}$

Stirling's Formula

$$n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}$$

Characteristic Functions

$$\phi(u) = E(e^{iux}) = \int e^{iux} f(x)\,dx \qquad i^{-n}\frac{d^n\phi}{du^n}\bigg|_{u=0} = \int x^n f(x)\,dx$$

Probability Distributions typically used $P(\{X = x\}) \equiv p(x)$

Binomial Distri.: $p(x) = \binom{n}{x} p^x q^{n-x} \qquad p + q = 1 \qquad E(x) = n\,p \qquad V(x) = n\,p\,q$

$\phi(u) = (q + p\, e^{iu})^n$: Characteristic Function

Poisson Distri.: $p(x) = \dfrac{\mu^x e^{-\mu}}{x!}$ $\qquad E(x) = \mu \qquad V(x) = \mu$

$\phi(u) = \exp\left[\mu(e^{iu} - 1)\right]$

Normal Distri.: $p(x) = \dfrac{1}{s\sqrt{2\pi}} \exp\left\{-\dfrac{1}{2s^2}(x-\mu)^2\right\}$ $\qquad E(x) = \mu \qquad V(x) = s^2$

$\phi(u) = \exp\left(i\mu u - \dfrac{1}{2}s^2u^2\right) \qquad \mathrm{erf}(a) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^a e^{-u^2}du$

χ^2 Distribution

$Z \sim N(0,1)$ normal distribution with mean of 0 and variance of 1

($x \sim X$: a random variable x is chosen from a distribution X)

$$W = \sum_{i=1}^{n} Z_i^2 \sim \chi^2\text{-distribution with } n \text{ degrees of freedom}$$

$$E(W) = n \qquad V(W) = 2n$$

X_i : random sample chosen from $N(\mu, s)$

$$W = \frac{1}{s^2}\sum_{i=1}^{n} (X_i - \bar{X})^2 \sim \chi^2 \text{ dist. w/ } n-1 \text{ d.o.f.}$$

$$S^2 = \frac{s^2W}{n-1} \qquad E(S^2) = \frac{s^2}{n-1}E(W) = s^2$$

$$p(\chi^2; n) = \frac{1}{2^{n/2}\Gamma(n/2)}(\chi^2)^{n/2-1}\exp\left(-\frac{\chi^2}{2}\right)$$

F-Distribution

χ_1^2, $\chi_2^2 \sim \chi^2$ distributions with n_1 and n_2 d.o.f., respectively, and are independant

$$F = \frac{\chi_1^2/n_1}{\chi_2^2/n_2} \sim F\text{-distribution with } n_1,\, n_2 \text{ degree of freedoms}$$

$$p(F; n_1, n_2) = \frac{n_1^{n_1/2} n_2^{n_2/2}}{B(n_1/2, n_2/2)} \frac{F^{n_1/2-1}}{(n_1F + n_2)^{(n_1+n_2)/2}}$$

$$\text{where } B(x,y) \equiv \int_0^1 t^{x-1}(1-t)^{y-1}dt \text{ (Beta function)}$$

Random sampling independently chosen from two normal populations with the same variance

M,N: sample sizes $\qquad U^2$,V^2: sample variances

$$F = \frac{U^2}{V^2} \sim F\text{-dist. w/ } M-1,\, N-1 \text{ d.o.f.}$$

Sample of size N from normal population with mean of μ

$\bar{X}$: sample mean $\qquad S^2$: sample variance

$$F = \frac{(\bar{X}-\mu)^2N}{S^2} \sim F\text{-dist. w/ } 1,\, N-1 \text{ d.o.f.}$$

Confidence Interval of Population Mean of Normal Distribution

$$F = \frac{(\bar{X}-\mu)^2N}{S^2} \sim F\text{-dist. w/ } 1,\, N-1 \text{ d.o.f.}$$

95% Confidence Interval

$$\bar{X} - S\sqrt{\frac{F_0}{N}} < \mu < \bar{X} + S\sqrt{\frac{F_0}{N}} \qquad P(F < F_0) = 0.95$$

Confidence Interval of Population Variance of Normal Distribution

$$\chi^2 = \frac{(N-1)S^2}{s^2} \sim \chi^2\text{-dist. w/ } N-1 \text{ d.o.f.}$$

90% Confidence Interval

$$\frac{(N-1)S^2}{\chi_0^2} < s^2 < \frac{(N-1)S^2}{\chi_1^2} \qquad P(\chi_0^2 > \chi^2 > \chi_1^2) = 0.90$$

Test about Mean of Normal Distribution

Comparison Between Two Population Means

Samples with size M and N chosen from two normal distributions with the same population variance

$\bar{X}$, $\bar{Y}$: sample means $\quad$ S_X^2, S_Y^2: sample variances

$$F = \frac{(\bar{X} - \bar{Y} - a)^2}{(M-1)S_X^2 + (N-1)S_Y^2} \cdot \frac{MN(M+N-2)}{M+N} \sim F\text{-dist. w/ } 1,\ M+N-2 \text{ d.o.f.}$$

where we let $\mu_X = \mu_Y = a$

Comparison Between Two Population Variances

M and N samples chosen from two normal distributions

S_X^2, S_Y^2: Respective sampling variances

$$F = \frac{S_X^2}{S_Y^2} \sim F\text{-dist. w/ } M-1,\ N-1 \text{ d.o.f.}$$

Propagation of Errors

$$f = f(x, y) \Longrightarrow \sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\sigma_{xy}$$

$$h = \prod_k f_k^{p_k} \Longrightarrow \frac{\Delta h}{h} = \sqrt{\sum_k \left(p_k \cdot \frac{\Delta f_k}{f_k}\right)^2}$$

Combination

$$\binom{n}{r} \equiv \frac{n!}{(n-r)!r!}$$

Bayes' Theorem

$$A_1 + A_2 + \cdots + A_n = \Omega : \text{Sample space}$$

$$P(A_i|E) = \frac{P(A_i)P(E|A_i)}{P(A_1)P(E|A_1) + \cdots + P(A_n)P(E|A_n)} \qquad {}^\forall E\text{: event}$$

where $P(A_i)$ is called the *prior probability*, and $P(A_i|E)$ the *posterior probability*