Notation

x, y: random variables

$$E(x) \equiv \lim_{\nu \to \infty} \left(\sum_{i=1}^{\nu} x_i/\nu \right) \qquad E(a\,x + b\,y) = a\,E(x) + b\,E(y) \qquad a,\ b:\ \text{constants}$$

$$V(x) \equiv E[\{x - E(x)\}^2] = E(x^2) - E^2(x) \qquad V(a\,x + b) = a^2 V(x)$$

x and y are independant: $\qquad E(x\,y) = E(x)E(y) \qquad V(x+y) = V(x) + V(y)$

x_i: Element in population $\quad \bar{x}$: Mean of population $\quad \sigma$: Standard deviation of population

$$\bar{x} = E(x) \qquad \sigma^2 = E\{(x - \bar{x})^2\} = E(x^2) - \bar{x}^{\,2}$$

$$\bar{X} \equiv \frac{1}{N}\sum_{i=1}^{N} X_i : \text{Sample mean} \qquad E(\bar{X}) = \bar{x} \qquad \begin{array}{l} N: \text{Sample size} \\ X_i: i\text{-th element in sample} \end{array}$$

$$S^2 \equiv \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})^2 = \frac{1}{N-1}\left(\sum_{i=1}^{N} X_i^2 - N\bar{X}^2 \right) : \text{Sample variance} \qquad E(S^2) = \sigma^2$$

$$E(X_i) = \bar{x} \qquad\qquad E(X_i^2) = E(x^2)$$

$$E(X_i\bar{X}) = \frac{1}{N}E\left\{ X_i\left(X_i + \sum_{k \neq i} X_k \right) \right\} = \frac{1}{N}\left\{ E(X_i^2) + \sum_{k \neq i} E(X_i X_k) \right\}$$

$$= \frac{1}{N}E(x^2) + \frac{N-1}{N}\bar{x}^{\,2}$$

$$E(\bar{X} \cdot \bar{X}) = E\left(\frac{1}{N}\sum_{k} X_k\bar{X} \right) = E(X_k\bar{X})$$

$$E\{(X_i - \bar{X})^2\} = E(X_i^2 - 2X_i\bar{X} + \bar{X}^2) = \frac{N-1}{N}\{E(x^2) - \bar{x}^{\,2}\} = \frac{N-1}{N}\sigma^2$$

Covariance and Correlation Coefficient of $x\,y$

$$\sigma_{xy} \equiv \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{N}\left\{ \left(\sum_{i=1}^{N} x_i y_i \right) - N\bar{x}\bar{y} \right\} \xrightarrow{N \to \infty} E(xy) - E(x)E(y)$$

$$\rho_{xy} \equiv \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

$$V(x+y) = V(x) + V(y) + 2\{E(xy) - E(x)E(y)\} = \sigma_x^2 + 2\sigma_{xy} + \sigma_y^2$$

For $q = f(p_i) \qquad \delta q = \sum_i \frac{\partial f}{\partial p_i}\delta p_i$

$$V(\delta q) = V\left(\sum_i \frac{\partial f}{\partial p_i}\delta p_i \right) = \sum_{i,j}(\partial_i f)\sigma_{ij}(\partial_j f)$$

$$\partial_i f \equiv \frac{\partial f}{\partial p_i} \qquad \sigma_{ij} \equiv E(\delta p_i \delta p_j) \qquad \sigma_{ij} : \text{Covariance matrix}$$

Change of Variables

Change of random variable $\boldsymbol{D} \subseteq \Re^n \to \boldsymbol{D}' \subseteq \Re^n$

$$X_i \in \boldsymbol{D} \longmapsto Y_i = f_i(X_1, \cdots, X_n) \in \boldsymbol{D}'$$

Relation between probability density function $p(X_1, \cdots, X_n)$ and $q(Y_1, \cdots, Y_n)$

$$p(X_1, \cdots, X_n) = \left| \frac{\partial(Y_1, \cdots, Y_n)}{\partial(X_1, \cdots, X_n)} \right| q(Y_1, \cdots, Y_n)$$

where we let the determinant non-zero everywhere $\longrightarrow$ One-to-one between $\boldsymbol{D}$ and $\boldsymbol{D'}$

Stirling's Formula

$$n! \sim \sqrt{2\pi} \, n^{n+\frac{1}{2}} e^{-n}$$

Characteristic Functions

$$\phi(u) = E(e^{iux}) = \int e^{iux} f(x)\, dx \qquad\qquad i^{-n} \frac{d^n \phi}{du^n} \bigg|_{u=0} = \int x^n f(x)\, dx$$

Probability Distributions typically used $P(\{X = x\}) \equiv p(x)$

Binomial Distri.: $\quad p(x) = \begin{pmatrix} n \\ x \end{pmatrix} p^x q^{n-x} \qquad p + q = 1 \qquad E(x) = n\,p \qquad V(x) = n\,p\,q$

$$\phi(u) = (q + p\,e^{iu})^n \; : \; \text{Characteristic Function}$$

Poisson Distri.: $\quad p(x) = \dfrac{\mu^x e^{-\mu}}{x!} \qquad\qquad E(x) = \mu \qquad V(x) = \mu$

$$\phi(u) = \exp\left[\mu(e^{iu} - 1) \right]$$

Normal Distri.: $\quad p(x) = \dfrac{1}{s\sqrt{2\pi}} \exp\left\{ -\dfrac{1}{2s^2}(x - \mu)^2 \right\} \qquad E(x) = \mu \qquad V(x) = s^2$

$$\phi(u) = \exp\left(i\mu u - \frac{1}{2} s^2 u^2 \right) \qquad\qquad \text{erf}(a) = \frac{2}{\sqrt{\pi}} \int_0^a e^{-u^2} du$$

χ^2 Distribution

$Z \sim N(0,1)$ normal distribution with mean of 0 and variance of 1

$$(x \sim X : \text{a random variable } x \text{ is chosen from a distribution } X)$$

$$W = \sum_{i=1}^n Z_i^2 \sim \chi^2\text{-distribution with } n \text{ degrees of freedom}$$

$$E(W) = n \qquad\qquad V(W) = 2n$$

X_i : random sample chosen from $N(\mu, s)$

$$W = \frac{1}{s^2} \sum_{i=1}^n (X_i - \bar{X})^2 \sim \chi^2 \text{ dist. w/ } n-1 \text{ d.o.f.}$$

$$S^2 = \frac{s^2 W}{n-1} \qquad\qquad E(S^2) = \frac{s^2}{n-1} E(W) = s^2$$

$$p(\chi^2; n) = \frac{1}{2^{n/2} \Gamma(n/2)} (\chi^2)^{n/2 - 1} \exp\left(-\frac{\chi^2}{2} \right)$$

F-Distribution

$\chi_1^2, \chi_2^2 \sim \chi^2$ distributions with n_1 and n_2 d.o.f., respectively, and are independant

$$F = \frac{\chi_1^2 / n_1}{\chi_2^2 / n_2} \sim F\text{-distribution with } n_1,\, n_2 \text{ degree of freedoms}$$

$$p(F; n_1, n_2) = \frac{n_1^{n_1/2} n_2^{n_2/2}}{B(n_1/2, n_2/2)} \frac{F^{n_1/2 - 1}}{(n_1 F + n_2)^{(n_1 + n_2)/2}}$$

$$\text{where } B(x, y) \equiv \int_0^1 t^{x-1}(1-t)^{y-1} dt \text{ (Beta function)}$$

Random sampling independently chosen from two normal populations with the same variance

M,N: sample sizes $\qquad$ U^2,V^2: sample variances

$$F = \frac{U^2}{V^2} \sim F\text{-dist. w/ } M-1,\ N-1 \text{ d.o.f.}$$

Sample of size N from normal population with mean of μ

$\bar{X}$: sample mean $\qquad$ S^2: sample variance

$$F = \frac{(\bar{X} - \mu)^2 N}{S^2} \sim F\text{-dist. w/ } 1,\ N-1 \text{ d.o.f.}$$

Confidence Interval of Population Mean of Normal Distribution

$$F = \frac{(\bar{X} - \mu)^2 N}{S^2} \sim F\text{-dist. w/ } 1,\ N-1 \text{ d.o.f.}$$

95% Confidence Interval

$$\bar{X} - S\sqrt{\frac{F_0}{N}} < \mu < \bar{X} + S\sqrt{\frac{F_0}{N}} \qquad P(F < F_0) = 0.95$$

Confidence Interval of Population Variance of Normal Distribution

$$\chi^2 = \frac{(N-1)S^2}{s^2} \sim \chi^2\text{-dist. w/ } N-1 \text{ d.o.f.}$$

90% Confidence Interval

$$\frac{(N-1)S^2}{\chi_0^2} < s^2 < \frac{(N-1)S^2}{\chi_1^2} \qquad P(\chi_0^2 > \chi^2 > \chi_1^2) = 0.90$$

Test about Mean of Normal Distribution

Comparison Between Two Population Means

Samples with size M and N chosen from two normal distributions with the same population variance

$\bar{X}, \bar{Y}$: sample means $\qquad$ S_X^2, S_Y^2: sample variances

$$F = \frac{(\bar{X} - \bar{Y} - a)^2}{(M-1)S_X^2 + (N-1)S_Y^2} \cdot \frac{MN(M+N-2)}{M+N} \sim F\text{-dist. w/ } 1,\ M+N-2 \text{ d.o.f.}$$

where we let $\mu_X = \mu_Y = a$

Comparison Between Two Population Variances

M and N samples chosen from two normal distributions

S_X^2, S_Y^2: Respective sampling variances

$$F = \frac{S_X^2}{S_Y^2} \sim F\text{-dist. w/ } M-1,\ N-1 \text{ d.o.f.}$$

Propagation of Errors

$$f = f(x, y) \implies \sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\sigma_{xy}$$

$$h = \prod_k f_k^{p_k} \implies \frac{\Delta h}{h} = \sqrt{\sum_k \left(p_k \cdot \frac{\Delta f_k}{f_k}\right)^2}$$

Combination

$$\binom{n}{r} \equiv \frac{n!}{(n-r)!r!}$$

Bayes' Theorem

$$A_1 + A_2 + \cdots + A_n = \Omega : \text{Sample space}$$

$$P(A_i|E) = \frac{P(A_i)P(E|A_i)}{P(A_1)P(E|A_1) + \cdots + P(A_n)P(E|A_n)} \qquad {}^{\forall}E: \text{event}$$

where $P(A_i)$ is called the *prior probability*, and $P(A_i|E)$ the *posterior probability*